Topology Vol. 11, pp. 265–270. Pergamon Press, 1972. Printed in Great Britain

ON THE COHOMOLOGY OF NASH SHEAVES

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(*Received* 10 *June* 1971)

§1. INTRODUCTION

NASH introduced in [4] a concept of real algebraic manifold, and in [1], Artin and Mazur made precise the appropriate category. For definitions and examples, see [1] and [3]. These structures, which will be called Nash manifolds in this paper, occupy an intermediate position between real algebraic varieties and differentiable manifolds; in particular they sometimes allow the use of algebraic techniques in differential topology. For a highly successful example, see [1].

It had been hoped that the techniques of sheaf cohomology, which have proved so powerful in algebraic geometry, could be applied. The object of this paper is to show that this is not the case.

Indeed, the only reasonable known method of computing sheaf cohomology is the Čech construction, and because of the direct limit involved, it is essential to have open coverings by cohomologically trivial subsets; Stein manifolds play this part in complex analytic geometry, and affine schemes in algebraic geometry. The obvious candidates in the Nash category would be open balls with their canonical structure. It is easy to see that if any cohomologically trivial subsets of an arbitrary Nash manifold exist, then these open sets must be among them. Unfortunately, they are not. We shall show that on an open interval there are Čech cocycles which are not coboundaries.

In fact, consider the open interval $(-1-\varepsilon, +1+\varepsilon)$ for some positive ε. $\{(-1-\varepsilon, +1), (-1, +1+\varepsilon)\}$ is an open cover, and a 1-cocycle for this cover is an algebraic function f on $(-1, +1) = (-1-\varepsilon, +1) \cap (-1, 1+\varepsilon)$; $+\sqrt{1-x^2}$ would be an example of such a cocycle. It is a coboundary if there are algebraic functions g_1 and g_2 defined on $(-1, 1+\varepsilon)$ and $(-1-\varepsilon, +1)$ respectively, such that on $(-1, +1)$, $g_1 - g_2 = f$. Now analytic functions satisfying these properties are easy to construct, and this is the key point of the proof of Cartan's Theorem B (see [2], especially exposé 17).

Extend f to some neighborhood in the complex plane, and choose an arc C and sub-arcs C_1 and C_2 as shown in Fig. 1, so that $C = C_1 - C_2$. If we define

$$g_i(z) = \frac{1}{2\pi} \int_{C_i} \frac{f(\zeta)}{\zeta - z}\, d\zeta,\ i = 1, 2,$$

the integral for g_1 certainly converges on $(-\frac{1}{2}, +\infty)$ and g_1 can be extended to $(-1, +\infty)$ by expanding C. Similarly, g_2 can be defined on $(-\infty, +1)$. Moreover,

$$g_1(z) - g_2(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z).$$

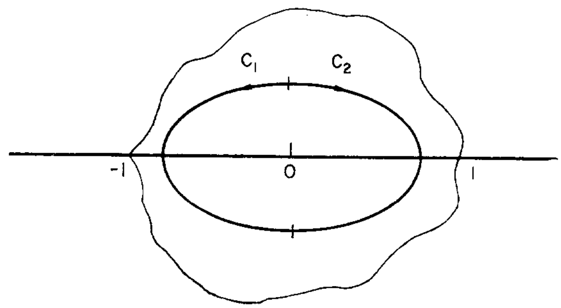

FIG. 1

However, the functions g_1 and g_2 defined in this way will in general not be algebraic, and we shall construct algebraic functions f such that corresponding analytic g_i's cannot be chosen algebraic. In particular, we shall show that $+\sqrt{1-x^2}$ is such a function.

My thanks go to Barry Mazur, who introduced me to the subject of Nash manifolds, and to Adrien Douady without whose help this example would never have been found.

§2. NOTATION

Let V be an open simply connected neighborhood of $[-1, +1]$, and let V_1 and V_2 be open subsets of V such that:

V_1, V_2 and $V_1 \cap V_2$ are simply connected and connected, $(-1, +1] \subset V_1$ and $[-1, +1) \subset V_2$, $-1 \notin V_1$ and $+1 \notin V_2$.

Set $V_1 \cap V_2 = U$.
Let α (resp. β) be a loop in V_1 (resp. V_2) with base point 0, whose image in $\pi_1(V_1 - \{1\}, 0)$ (resp. $\pi_1(V_2 - \{-1\}, 0)$), also written α (resp. β), is a generator. Set $\gamma = \alpha\beta$.

A covering map $\pi\colon (X, x) \to (B, b)$ is determined by $\pi_*(\pi_1(X, x))$, and we shall describe covering spaces by the corresponding subgroups of the fundamental group of the base space. The fiber $\pi^{-1}(b)$ of π will be written $F(\pi)$. Similarly, we will describe ramified coverings by subgroups of the fundamental group of the base space with the branch locus removed.

Let S (see Fig. 2) be the covering of V ramified in $\{-1, +1\}$ corresponding to the normal subgroup of $\pi_1(V - \{-1, +1\}, 0)$ generated by α^2, β^2 and γ. Let $\pi_{S/V}$ denote the covering map $S \to V$. Let $\tilde{S}$ be the universal covering space of S, and $\pi_{\tilde{S}/S}$ be the projection. $\pi_{\tilde{S}/V} = \pi_{S/V} \circ \pi_{\tilde{S}/S}$ makes $\tilde{S}$ a ramified cover of V, corresponding to the normal subgroup of $\pi_1(V - \{-1, +1\}, 0)$ generated by α^2 and β^2.

Recall that if $\pi\colon (X, x) \to (B, b)$ is a covering space, there is a natural action of $\pi_1(B, b)$ on $F(\pi)$. If U is a simply connected and connected neighborhood of b, this action can be uniquely extended to $\pi^{-1}(U)$. Moreover, if $\pi_*(\pi_1(X, x))$ is a normal subgroup of $\pi_1(B, b)$, then this action can be extended to all of X, and in this case $\operatorname{Aut}_B(X) \cong \pi_1(B, b)/\pi_*(\pi_1(X, x))$. We will denote the action of $\alpha \in \pi_1(B, b)$ by $[\alpha]\colon F(\pi) \to F(\pi)$ (or $[\alpha]\colon X \to X$ if X is normal over B).

Choose $s \in \pi_{S/V}^{-1}(0)$, and $\tilde{s} \in \pi_{\tilde{S}/S}^{-1}(s)$. All of our coverings so far are normal. $\pi_1(S, s)$ is infinite cyclic, generated for instance by a lifting $\hat{\gamma}$ of γ to S, so $\operatorname{Aut}(\tilde{S}/S) \cong \mathbb{Z}$, and $[\hat{\gamma}]$ is a generator of $\operatorname{Aut}(\tilde{S}/S)$.

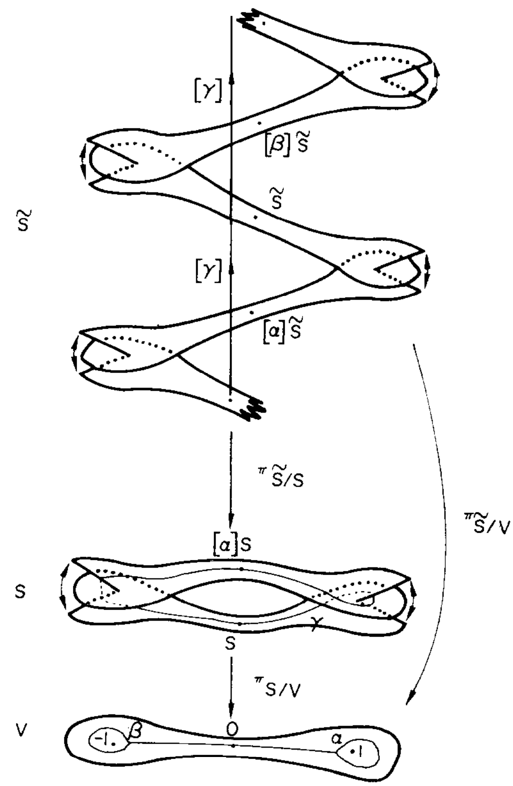

FIG. 2

$\mathrm{Aut}(S/V) \cong \mathbb{Z}/2\mathbb{Z}$, generated by $[\alpha]$ (or $[\beta]$, as $[\alpha] = [\beta]$ as automorphisms of S). $\mathrm{Aut}(\tilde{S}/V)$ is isomorphic to the non-trivial extension of $\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$; it is generated for instance by $[\alpha]$ and $[\gamma]$, with relations $[\alpha][\gamma] = [\gamma]^{-1}[\alpha]$, and $[\alpha]^2 = 1$. Note that as automorphisms of $\tilde{S}$, $[\gamma] = [\hat{\gamma}]$.

A covering map always has sections over any simply connected set in the base space, and such a section is unique if base points are to be preserved and the set is connected. Let φ (resp. θ) be the section of $\pi_{S/V}$ (resp. $\pi_{\tilde{S}/V}$) over U satisfying this condition, and let ψ be the section of $\pi_{\tilde{S}/S}$ over $\varphi(u)$, so that $\psi \circ \varphi = \theta$.

Let f be a meromorphic function on U, such that the function $\hat{f}$ on $\varphi(U)$ defined by $\hat{f}(p) = f(\varphi(p))$ can be extended to all of S. Let h be the function defined on S by $h(p) = f([\alpha](p)) - f(p)$, and $\tilde{h}$ be defined by $\tilde{h}(p) = h(\pi_{\tilde{S}/S}(p))$. Let $\tilde{f}$ be the function on $\tilde{S}$ defined by $\tilde{f}(p) = \hat{f}(\pi_{\tilde{S}/S}(p))$. Let g_1 (resp. g_2) be a meromorphic function defined on V_1 (resp. V_2) such that $f = g_1 | U - g_2 | U$.

§3. THE MAIN THEOREM

THEOREM 1. (a) *There exist unique meromorphic functions* $\tilde{g}_1$ *and* $\tilde{g}_2$ *on* $\tilde{S}$ *such that* $\tilde{g}_i(\theta(z)) = g_i(z)$, $i = 1, 2$

(b) *These functions satisfy*

(i) $\tilde{g}_i([\gamma]^n(p)) = \tilde{g}_i(p) + nh(p)$;

(ii) $\tilde{g}_1([\alpha](p)) = \tilde{g}_1(p)$; $\tilde{g}_2([\beta](p)) = \tilde{g}_2(p)$;

(iii) $\tilde{g}_1(p) - \tilde{g}_2(p) = f(p)$.

Proof. (a) The uniqueness of the g_i is trivial, as $\tilde{S}$ is connected, and the values of the g_i's are given on an open set. The existence follows easily from analytic continuation along curves, but in order to prove (b), we need a more precise description.

LEMMA 1. *Set* $S_i = \pi_{S/V}^{-1}(V_i)$, *and* $\tilde{S}_i = \pi_{\tilde{S}/V}^{-1}(V_i)$, $i = 1, 2$.
(a) *There exist unique sections* $\psi_i : S_i \to \tilde{S}$ *of* $\pi_{\tilde{S}/S}$ *which extend* ψ.
(b) *On* $\pi_{S/V}^{-1}(U) = \varphi(U) \cup [\alpha](\varphi(U))$, *these sections are both defined, and satisfy*

$$\psi_1(p) = \psi_2(p) \text{ for } p \in \varphi(U)$$
$$\psi_1(p) = [\gamma](\psi_2(p)) \text{ for } p \in [\alpha](\varphi(u)).$$

Proof of Lemma. S_1 and S_2 are simply connected, so the ψ_i exist, and they agree on $\varphi(U)$ by definition as they extend ψ. Thus it suffices to prove the second part of (b), that is to show $\psi_1([\alpha]s) = [\gamma]\psi_2([\alpha]s)$. Let $\hat{\alpha}$ be the lifting of α to S such that $\hat{\alpha}(0) = s$; then $\hat{\alpha}(1) = [\alpha](s)$. $\psi_1 \circ \hat{\alpha}$ is the lifting $\tilde{\alpha}$ of α to $\tilde{S}$ such that $\tilde{\alpha}(0) = \tilde{s}$, so $[\alpha](\tilde{s}) = \tilde{\alpha}(1) = \psi_1 \circ \hat{\alpha}(1) = \psi_1([\alpha](s))$. Similarly, $[\beta](\tilde{s}) = \psi_2([\beta](s))$, so $\psi_1([\alpha](s)) = [\alpha](\tilde{s}) = [\alpha][\beta^{-1}\beta]\tilde{s} = [\alpha\beta^{-1}][\beta](\tilde{s}) = [\gamma]\psi_2([\beta](s)) = [\gamma](\psi_2[\alpha](s))$.

Proof of Theorem (continued). For all $p \in \tilde{S}_i$, there exists a unique integer n_i such that $p = [\gamma]^{n_i}\psi_i(\pi_{\tilde{S}/S}(p))$, since $\tilde{S}$ is an infinite cyclic covering of S. Let $\hat{g}_i$ be the function on S_i given by $\hat{g}_i(p) = g_i(\pi_{S/V}(p))$, and define $\tilde{g}_i$ on $\tilde{S}_i$ by

$$\tilde{g}_i(p) = \hat{g}_i(\pi_{\tilde{S}/S}(p)) + n_i\tilde{h}(p).$$

LEMMA 2. *On* $\tilde{S}_1 \cap \tilde{S}_2$, *both* $\tilde{g}_1$ *and* $\tilde{g}_2$ *are defined, and they satisfy* $\tilde{g}_1(p) - \tilde{g}_2(p) = \tilde{f}(p)$.
Proof of Lemma. If $p \in \tilde{S}_1 \cap \tilde{S}_2$, then

$$p = [\gamma]^{n_1}\psi_1(\pi_{\tilde{S}/S}(p)) = [\gamma]^{n_2}\psi_2(\pi_{\tilde{S}/S}(p)),$$

and there are two cases to consider.

(i) $\pi_{\tilde{S}/S}(p) \in \varphi(U)$. Then $n_1 = n_2$, and $g_1(p) - g_2(p) = \hat{g}_1(\pi_{\tilde{S}/S}(p)) + n_1\tilde{h}(p) - \hat{g}_2(\pi_{\tilde{S}/S}(p)) - n_2\tilde{h}(p) = \hat{g}_1(\pi_{\tilde{S}/S}(p)) - \hat{g}_2(\pi_{\tilde{S}/S}(p)) = \hat{f}(\pi_{\tilde{S}/S}(p)) = \tilde{f}(p)$.

(ii) $\pi_{\tilde{S}/S}(p) \in [\alpha](\varphi(U))$. Then $n_2 = n_1 + 1$, by Lemma 1, and $\tilde{g}_1(p) - \tilde{g}_2(p) = \hat{g}_1(\pi_{\tilde{S}/S}(p)) + n_1\tilde{h}(p) - \hat{g}_2(\pi_{\tilde{S}/S}(p)) - (n+1)\tilde{h}(p) = \hat{f}([\alpha](\pi_{\tilde{S}/S}(p))) - \tilde{h}(p) = \hat{f}([\alpha](\pi_{\tilde{S}/S}(p))) - (\hat{f}([\alpha]\pi_{\tilde{S}/S}(p)) - \hat{f}(\pi_{\tilde{S}/S}(p))) = \hat{f}(\pi_{\tilde{S}/S}(p)) = \tilde{f}(p)$.

Proof of Theorem (Continued and ended). By Lemma 2, we can extend $\tilde{g}_1$ by setting $g_1(p) = \tilde{f}(p) + \tilde{g}_2(p)$, for $p \in \tilde{S}_2$, and similarly $\tilde{g}_2(p) = \tilde{g}_1(p) - \tilde{f}(p)$ for $p \in \tilde{S}_1$. Moreover, part (b), (iii) of the theorem is proved. To see that $\tilde{g}_i \circ [\gamma]^n = \tilde{g}_i + n\tilde{h}$, note that both sides are meromorphic functions on $\tilde{S}$, and that they agree on $\theta(U)$ by the definition of $\tilde{g}_i$, so that as $\tilde{S}$ is connected, they agree everywhere.

For (b)(ii), recall that by definition $\hat{g}_1 \circ [\alpha] = \hat{g}_1$, and that if $p \in \psi_1(S_1)$, then $[\alpha]\psi_1(p) = \psi_1([\alpha]p)$. Therefore, as $\hat{g}_1(\psi_1(p)) = \hat{g}_1(p) = \hat{g}_1([\alpha]p) = \tilde{g}_1(\psi_1([\alpha]p)) = \tilde{g}_1([\alpha]\psi_1(p))$, $\tilde{g}_1$ and $\tilde{g}_1 \circ [\alpha]$ are meromorphic functions on $\tilde{S}$ which agree on $\psi_1(S_1)$. So they agree everywhere.

Q.E.D.

We have now defined the functions g_i on $\tilde{S}$, and for any point $p \in \tilde{s}$, we have proved that the values of $\tilde{g}_i$ at the points $[\gamma]^n(p)$ form an arithmetic progression. Therefore if the reason of the progression is not zero, we know that the g_i cannot be defined on any quotient of $\tilde{S}$ finite over V. We shall in fact give a more precise result.

Consider $[\alpha]: \tilde{S} \to \tilde{S}$, which is an automorphism of $\tilde{S}$ over V (not over S!). We may consider the quotient $\tilde{S}_\alpha$ of $\tilde{S}$ by $[\alpha]$, (see Fig. 3) whose points are pairs $\{p, [\alpha]p\}$ of points of $\tilde{S}$: the map $\pi_{S/S_\alpha}: \tilde{S} \to \tilde{S}_\alpha, p \mapsto \{p, [\alpha]p\}$ is a ramified covering map. A similar construction of $\tilde{S}_\beta$, $\pi_{\tilde{S}/\tilde{S}_\beta}$ can be carried out. As $\tilde{g}_1 = \tilde{g}_1 \circ [\alpha]$, there exists a function $\bar{g}_1$ on S_1 defined by $\bar{g}_1(\pi_{\tilde{S}/\tilde{S}_\alpha}(p)) = \tilde{g}_1(p)$; define similarly a function $\bar{g}_2$ on S_β.

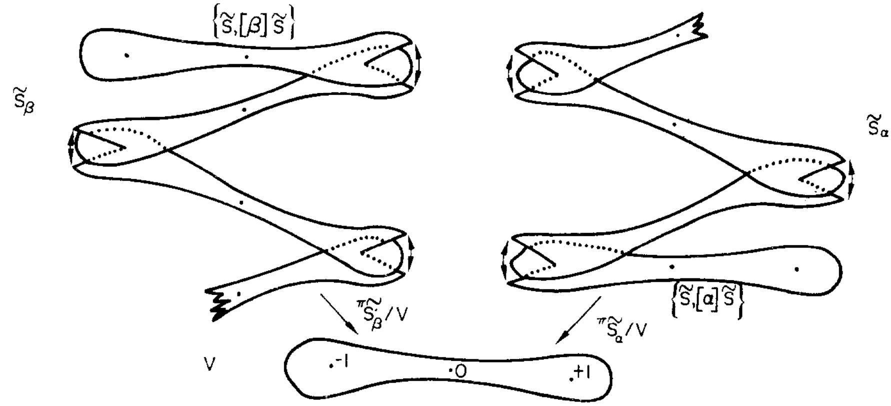

FIG. 3

THEOREM 2. *If the Riemann surface of f is S (i.e. if h does not vanish identically), then S_α is the Riemann surface of g_1, and S_β is the Riemann surface of g_2.*

Proof. By changing the base point if necessary, we may assume that $h(0) \neq 0, \infty$. We shall show that $\bar{g}_1$ separates the points of the fiber $F(\pi_{\tilde{S}_\alpha/V})$. This follows from the facts that $\tilde{g}_1$ separates the points of $F(\pi_{\tilde{S}/S})$, as we noted above, and that the map $\pi_{\tilde{S}/\tilde{S}_\alpha} | F(\pi_{\tilde{S}/S}): F(\pi_{\tilde{S}/S}) \to F(\pi_{S_\alpha/V})$ is bijective. To show this, we shall construct an inverse. A point of $F(\pi_{\tilde{S}_\alpha/V})$ is a pair $\{p, [\alpha]p\}$, $p \in \pi_{\tilde{S}/V}^{-1}(0)$, and since $\pi_{\tilde{S}/S}(\{p, [\alpha]p\}) = \{s, [\alpha]s\}$, exactly one element of $\{p, [\alpha]p\}$ lies in $F(\pi_{\tilde{S}/S})$. The map associating this element to the pair $\{p, [\alpha]p\}$ is the desired inverse. *Q.E.D.*

COROLLARY 1. *If f, g_1 and g_2 are as in Theorem 2, then g_1 and g_2 cannot be algebraic.*

COROLLARY 2. *The cocycle $\sqrt{1 - x^2}$ on $\mathbb{R}$, for the covering $\{(-\infty, 1), (-1, +\infty)\}$, represents a nontrivial class in the first cohomology group with values in the sheaf of germs of algebraic functions.*

§4. THE MULTIPLICATIVE THEORY

There is a parallel multiplicative theory. We shall use the same set of Vs, Ss, fs and gs as before, except that now $f = g_1/g_2$ on U, and $h = [\alpha]f/f$.

THEOREM 1′. (a) *There exist functions g_1 and g_2 on $\tilde{S}$ normalized as in Theorem* 1. (b) *These functions satisfy the following relations*

(i) $g_i \circ [\gamma]^n = g_i h^n$
(ii) $\tilde{g}_1 \circ [\alpha] = \tilde{g}_1, \tilde{g}_2 \circ [\beta] = \tilde{g}_2$
(iii) $\tilde{f} = \tilde{g}_1/\tilde{g}_2$.

The proof is identical to that of Theorem 1.

THEOREM 2′. *Let f, g_1 and g_2 be as in Theorem 1′, and suppose that h is not identically either 1 or -1. Then the Riemann surface of g_1 is S_α and the Riemann surface of g_2 is S_β.*

The reason for which h cannot be allowed to be identically -1 is that the values on the fiber $F(\pi_{\bar{S}/S})$ now form a geometric progression rather than an arithmetic progression, and a geometric progression of reason -1 is periodic, so the argument about separating points fails. Other than this observation, the proofs are identical.

Example. $+\sqrt{1-x^2} = \sqrt{1+x}\left(\frac{1}{\sqrt{1-x}}\right)^{-1}$ shows the necessity of excluding the case $h = -1$.

COROLLARY 1′. *If f, g_1, g_2, and h are as in Theorem 2′, then g_1 and g_2 are not algebraic.*

COROLLARY 2′. *If f is as in Theorem 2′, and besides does not vanish on $(-1, +1)$ then f defines a non-trivial cocycle in the sheaf of invertible germs of algebraic functions. For instance, $2+\sqrt{1-x^2}$ represents a nonzero multiplicative cohomology class.*

COROLLARY 3. *The Nash-locally trivial line bundle obtained over $\mathbb{R}$ by taking a trivial line bundle over $(-\infty, +1)$ and another over $(-1, +\infty)$, and by gluing the section identically 1 of the first to the section $2+\sqrt{1-x^2}$ defined over $(-1, 1)$ of the second, is not a trivial bundle. In fact it is not embeddable.*

Indeed, it is easy to show that an embeddable bundle over $\mathbb{R}$ is trivial.

The non-trivial cocycles in Theorems 1 and 1′ have compact support, and they can therefore be used to construct non-trivial bundles over compact Nash manifolds, even embeddable ones.

COROLLARY 4. *Let $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the canonical circle, and take $f\colon (-\frac{5}{8}, \frac{5}{8}) \to S$, $f(t) = (\cos 2\pi t, \sin 2\pi t)$ to be a covering consisting of a single piece. A Nash cocycle with respect to this covering is a function on the self intersection of f with itself, which is just the set $\{(x, y) \mid x < 0, |y| < \frac{\sqrt{2}}{2}\}$. Consider the function $2+\sqrt{\frac{1}{2}-y^2}$. It is a multiplicative Nash cocycle which is not a Nash coboundary, as the map $(x, y) \mapsto y$ induces an isomorphism between its restriction to the set $\{(x, y) \in S \mid x < 0\}$ and the example in Corollary 3.*

Therefore the associated line bundle over the canonical circle, which is topologically trivial as $2+\sqrt{\frac{1}{2}-y^2}$ is positive, is Nash-locally trivial but not trivial.

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