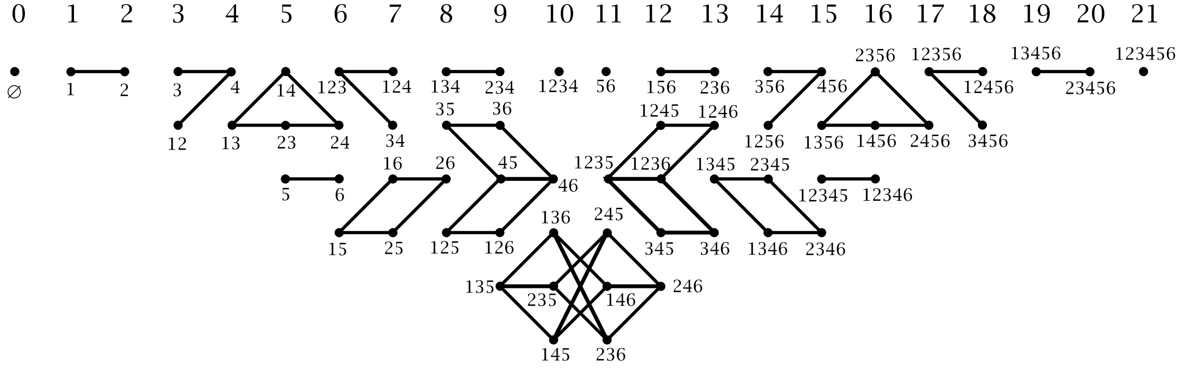
The special orthogonal group $SO(n)$ is high on the list of important topological spaces, yet its homology and cohomology exhibit some surprising subtleties. The complications arise from the presence of torsion in the integer homology and cohomology, but fortunately the torsion consists just of elements of order 2. Both the integer cohomology ring modulo torsion and the mod 2 cohomology ring have structures that are easy to describe (see Section 3D of my book):

(1) $H^*(SO(n);\mathbb{Z})$ modulo torsion is the exterior algebra on generators $a_3, a_7, \cdots, a_{4k-1}$ for $n = 2k+1$ and $a_3, a_7, \cdots, a_{4k-1}, a'_{2k+1}$ for $n = 2k+2$. Here subscripts denote degrees, so $a_i \in H^i$ and $a'_{2k+1} \in H^{2k+1}$.

(2) $H^*(SO(n);\mathbb{Z}_2)$ is the polynomial algebra on generators $b_i$ of odd degree $i < n$, truncated by the relations $b_i^{p_i} = 0$ where $p_i$ is the smallest power of 2 such that $b_i^{p_i}$ has degree $\geq n$.

The subtleties arise when one tries to describe the actual integral cohomology ring itself. In principle this follows from a calculation of mod 2 Bockstein homomorphisms, which is not difficult and is described in Example 3E.7 of my book. The cases of $SO(5)$ and $SO(7)$ are worked out in detail there. Here's what the Bocksteins look like for $SO(7)$:



The numbers across the top of the figure denote degrees. Each dot in the $i$th column represents a basis element for $H^i(SO(7);\mathbb{Z}_2)$ viewed as a vector space over $\mathbb{Z}_2$, with the label on the dot telling which class the dot represents. For example the dot labeled 234 is the product $b_2 b_3 b_4$, where the relations $b_{2i} = b_i^2$ allow the $b_i$'s with even subscripts to be expressed in terms of those with odd subscripts, the generators in statement (2) above. The line segments in the diagram indicate the nonzero Bocksteins. These are homomorphisms $\beta\colon H^i(SO(7);\mathbb{Z}_2) \to H^{i+1}(SO(7);\mathbb{Z}_2)$ satisying $\beta^2 = 0$. The nontorsion in $H^*(SO(7);\mathbb{Z})$ corresponds to $\operatorname{Ker}\beta/\operatorname{Im}\beta$, while the torsion elements correspond to $\operatorname{Im}\beta$. For example, the nontorsion element $a_3$ corresponds to $b_3 + b_1 b_2$ (these are $\mathbb{Z}_2$ classes so signs don't matter) and $a_7$ corresponds to either $b_1 b_2 b_4$ or $b_3 b_4$.

An additive basis for $H^*(SO(n);\mathbb{Z}_2)$ consists of the products $b_{i_1}\cdots b_{i_k}$ with $0 < i_1 < \cdots < i_k < n$. These classes are in one-to-one correspondence with the cells in a CW structure on $SO(n)$. There are $2^{n-1}$ of these classes, so the size of $H^*(SO(n);\mathbb{Z}_2)$ grows exponentially with $n$, in contrast with the dimension of $SO(n)$ which is $n(n-1)/2$, just quadratic in $n$. Thus the maximum size of the individual groups $H^i(SO(n);\mathbb{Z}_2)$ is also growing exponentially with $n$, although for fixed $i$ this group is independent of $n$ when $n > i+1$.

M. A. Agosto and J. J. Perez have written a Mathematica program to draw diagrams showing nonzero Bocksteins in $H^*(SO(n);\mathbb{Z}_2)$ for general $n$. In the range $5 \leq n \leq 12$ these are shown in the accompanying pdf files, with a different convention for displaying the picture than in the figure above, so that Poincaré duality appears as a 180 degree rotational symmetry about the center point of the diagram rather than as reflection across a vertical line.