TETHERS AND HOMOLOGY STABILITY FOR SURFACES

ALLEN HATCHER AND KAREN VOGMTANN

Abstract. Homological stability for sequences $G_n \to G_{n+1} \to \cdots$ of groups is often proved by studying the spectral sequence associated to the action of $G_n$ on a highly-connected simplicial complex whose stabilizers are related to $G_k$ for $k < n$. When $G_n$ is the mapping class group of a manifold, suitable simplicial complexes can be made using isotopy classes of various geometric objects in the manifold. In this paper we focus on the case of surfaces and show that by using more refined geometric objects consisting of pairs of dual curves together with arcs that tether these pairs to the boundary, the stabilizers can be greatly simplified and consequently also the spectral sequence argument. We give a careful exposition of this program and its basic tools, then illustrate the method using braid groups before treating mapping class groups of orientable surfaces in full detail.

Introduction

Many classical groups occur in sequences $G_n$ with natural inclusions $G_n \to G_{n+1}$. Examples include the symmetric groups $\Sigma_n$, linear groups such as $GL_n$, the braid groups $B_n$, mapping class groups $M_{n,1}$ of surfaces with one boundary component, and automorphism groups of free groups $Aut(F_n)$. A sequence of groups is said to be homologically stable if the natural inclusions induce isomorphisms on homology $H_i(G_n) \to H_i(G_{n+1})$ for $n$ sufficiently large with respect to $i$. All of the sequences of groups mentioned above are homologically stable. This terminology is also slightly abused when there is no natural inclusion, such as for the mapping class groups $M_n$ of closed surfaces and outer automorphism groups of free groups $Out(F_n)$; in this case, we say the series is homologically stable if the $i$th homology is independent of $n$, for $n$ sufficiently large with respect to $i$.

Homology stability is a very useful property. It sometimes allows one to deduce properties of the limit group $G_\infty = \lim_{\to} G_n$ from properties of the groups $G_n$; the classical example of this is Quillen’s proof that various $K$-groups are finitely generated. It is also useful in the opposite direction: it is sometimes possible to compute invariants of the limit group $G_\infty$, which by stability are invariants of the groups $G_n$; an example of this is the computation by Madsen and Weiss of the stable homology of the mapping class group. Finally, there is the obvious advantage that homology computations which are unmanageable for $n$ large can sometimes be done in $G_n$ for $n$ small.

In unpublished work from the 1970’s Quillen introduced a general method of proving stability theorems which was used by many authors in subsequent years (the earliest examples include [24], [23], [7], [22], and [10]). The idea is to find a highly-connected complex $X_n$ on which $G_n$ acts, such that stabilizers of simplices are isomorphic to $G_m$ for $m < n$. One then examines a slight variant of the equivariant homology spectral sequence for this action; this
has

\[ E^1_{p,q} = \begin{cases} H_q(G, \mathbb{Z}) & \text{for } p = -1 \\ \bigoplus_{\sigma \in \Sigma_p} H_q(\text{stab}(\sigma); \mathbb{Z}) & \text{for } p \geq 0 \end{cases} \]

where \( \Sigma_p \) is a set of representatives of orbits of \( p \)-simplices. The fact that \( X_n \) is highly connected implies that this spectral sequence converges to 0 for \( p + q \) small compared to \( n \), and the fact that simplex stabilizers are smaller groups \( G_m \) means that the map \( H_i(G_{n-1}) \to H_i(G_n) \) induced by inclusion occurs as a \( d^1 \) map in the spectral sequence. If one assumes the quotient \( X_n/G_n \) is highly connected and one or two small conditions of a more technical nature are satisfied, then an induction argument on \( i \) can be used to prove that this \( d^1 \) map is an isomorphism for \( n \) and \( i \) in the approximate range \( n > 2i \).

This is the ideal situation, but in practice the original proofs of homology stability were often more complicated because the complexes \( X_n \) chosen had simplex stabilizers that were not exactly the groups \( G_m \) for \( m < n \). For example, in the case of mapping class groups of surfaces the stabilizers were mapping class groups of surfaces with more than one boundary component, so one had to prove a secondary result that the homology in the stable range did not depend on the number of boundary components, necessitating a second spectral sequence argument involving another sequence of complexes \( X_n \).

For the groups \( \text{Aut}(F_n) \) and \( \text{Out}(F_n) \) a way to avoid the extra complications was developed in the papers [15] and [16] with further refinements and extensions in [17]. The idea was to use variations of the original complexes studied in earlier papers [13, 14] that included more data. In [15] this extra data consisted of supplementary 2-spheres in the ambient 3-manifold that were called “enveloping spheres”, while in [16] and [17] this extra data was reformulated in terms of arcs joining 2-spheres to basepoints in the boundary of the manifold. These arcs could be interpreted as “tethering” the spheres to the boundary.

In the present paper we show how this tethering idea can be used in the case of mapping class groups of surfaces. As above, tethers are arcs to a basepoint in the boundary, while at the other end they attach not to individual curves in the surface but to pairs of curves intersecting transversely in one point. More precisely (see Figure 1), for an orientable surface \( S = S_{g,s} \) of genus \( g \) with \( s \geq 0 \) boundary components a \textit{chain} is an ordered pair \( c = (a, b) \) of simple closed curves in \( S \) which intersect transversely in a single point. If \( \partial S \) is nonempty, a \textit{tether} for a chain \( c = (a, b) \) is an arc \( t \) embedded in \( S \) which joins a point of \( b - a \) to a fixed basepoint \( p \in \partial S \) and is otherwise disjoint from the chain. The key feature of a tethered chain is that cutting \( S \) open along a tethered chain produces a surface of one lower genus but the same number of boundary components.

A \textit{system of chains} is a set \( \{c_0, \ldots, c_k\} \) of disjoint chains. We form a simplicial complex \( Ch(S) \) with one \( k \)-simplex for each isotopy class of systems of \( k + 1 \) chains. Similarly, a \textit{system of tethered chains} is a set \( \{(c_0, t_0), \ldots, (c_k, t_k)\} \) of chains \( c_i \) with tethers \( t_i \), all these tethered chains being disjoint except at \( p \). We form a simplicial complex \( TCh(S) \) with one \( k \)-simplex for each isotopy class of systems of \( k + 1 \) tethered chains.

Our main new result is:

**Theorem.** The complexes \( Ch(S) \) and \( TCh(S) \) are \((g - 3)/2\)-connected.
The complex $TCh(S)$ is used for the spectral sequence showing that the homology group $H_i$ of the mapping class group of $S_{g,s}$ is independent of $g$ for fixed $s > 0$, in the stable dimension range $g > 2i + 1$. To extend this to the case $s = 0$ and to show that the homology is also independent of $s$ in the stable range we use the complex $Ch(S)$. (Even if one is interested only in closed surfaces it is necessary to consider the case of non-empty boundary in order to have a way to compare mapping class groups in different genus.)

The best stable dimension range that these simple sorts of spectral sequence arguments can yield has slope 2, as in the inequality $g > 2i + 1$. This is not the optimal range, which has slope $3/2$, arising from more intricate spectral sequence arguments. See [4, 20, 25] for details.

The complexes of chains and tethered chains that we show are highly connected have found other recent applications as well in [18] and [27]. In higher dimensions the natural analog of a tethered chain is a pair of $k$-spheres in a smooth manifold $M^{2k}$ intersecting transversely in a single point, together with an arc tethering one of the spheres to a basepoint in $\partial M$. These tethered sphere-pairs play a central role in recent work of Galatius and Randal-Williams [9] on homology stability for $BDiff(M)$ for certain manifolds $M^{2k}$ including the base case that $M$ is obtained from a connected sum of copies of $S^k \times S^k$ by deleting the interior of a $2k$-ball.

Here is an outline of the paper. In Section 1 we present the basic spectral sequence argument and in Section 2 we review the tools used prove the key connectivity results. In Section 3 we give a warm-up example illustrating the method in a particularly simple case, proving Arnold’s homology stability theorem for braid groups. In Sections 4 and 5 we reprove results about curve complexes and arc complexes due to Harer and Wahl that will be used in Section 6 to prove the main connectivity theorems for $Ch(S)$ and $TCh(S)$. Finally in Section 7 we deduce homology stability for mapping class groups.

**Remark.** A draft version of this paper dating from 2006 has been informally circulated for a number of years. Some of the results have been used by other authors since then, so we have decided to make this more polished version of that draft publicly available.

### 1. The basic spectral sequence argument

In this section we give the simplest form of the spectral sequence argument for proving homology stability of a sequence of group inclusions $\cdots \to G_n \to G_{n+1} \to G_{n+2} \to \cdots$. The input for the spectral sequence will be an action of $G_n$ on a complex $X_n$ for each $n$. To deduce stability we will need to assume:
(1) The action of $G_n$ is transitive on vertices of $X_n$.
(2) The stabilizer of a $p$-simplex fixes the simplex pointwise, and is conjugate to the image of $G_{n-r}$ in $G_n$ for some $r$ with $1 \leq r \leq p + 1$; in particular, the stabilizer of a vertex is conjugate to $G_{n-1}$.
(3) If $e$ is an edge of $X_n$ with vertices $v$ and $w$, then there is an element of $G_n$ taking $v$ to $w$ which commutes with all elements of the stabilizer of $e$.
(4) Both $X_n$ and the quotient $X_n/G_n$ are highly connected. In fact all that is needed is that their homology groups vanish up to some dimension, rather than their homotopy groups. However, for simplicity we will make statements in terms of connectivity.

The actual dimension range in which homology stability holds will depend on the connectivities of $X_n$ and $X_n/G_n$. The best result that the method can yield is that $H_i(G_{n-1}) \to H_i(G_n)$ is an isomorphism for $n > 2i - c$ and a surjection for $n = 2i - c$, for some constant $c$. Then it will just be a matter of determining $c$ in each case.

1.1. Construction of the spectral sequence. For $G = G_n$ let $E_q G$ be a free resolution of $Z$ by $Z[G]$-modules, and let

$$\cdots \to C_p \to C_{p-1} \to \cdots \to C_0 \to C_{-1} = Z \to 0$$

be the augmented chain complex of $X = X_n$. The action of $G$ on $X$ makes $C_*$ into a complex of $Z[G]$-modules, so we can take the tensor product over $Z[G]$ to form a double complex $C_* \otimes_G E_q G$. Filtering this double complex horizontally and then vertically or vice versa gives rise to two spectral sequences, both converging to the same thing.

Using the horizontal filtration, the $E^1_{p,q}$ term of the associated spectral sequence is formed by taking the $p$th homology of $C_* \otimes_G E_q G$. If we assume $X$ is highly connected, say $c(X)$-connected, then the complex $C_*$ is exact through dimension $c(X)$. Since $E_q G$ is free, $C_* \otimes_G E_q G$ is exact in the same range, so $E^2_{p,q} = 0$ for $p \leq c(X)$. In particular the spectral sequence converges to 0 in the range $p + q \leq c(X)$, so the same will be true for the other spectral sequence as well.

For the second spectral sequence, if we filter vertically instead of horizontally the associated $E^1_{p,q}$ term becomes $H_q(G;C_p)$. By Shapiro’s lemma (see, e.g. [6] p. 73) this reduces to

$$E^1_{p,q} = \bigoplus_{\sigma \in \Sigma_p} H_q(stab(\sigma);\mathbb{Z}/\sigma)$$

where $\Sigma_p$ is a set of representatives for orbits of $p$-simplices (if we consider a “$(-1)$-simplex” to be empty, with stabilizer all of $G$), and $\mathbb{Z}/\sigma$ is $\mathbb{Z}$ twisted by the orientation action of $stab(\sigma)$ on $\sigma$. In our case $stab(\sigma)$ is conjugate to $G_{n-r}$ for some $r = r(\sigma)$ and fixes $\sigma$, so $\mathbb{Z}/\sigma$ is an untwisted $\mathbb{Z}$ and the $E^1$ terms become simply

$$E^1_{p,q} = \bigoplus_{\sigma \in \Sigma_p} H_q(G_{n-r})$$

The $q$-th row of the $E^1$ page is the augmented chain complex of the quotient $X/G$ with coefficients in the system $\{H_q(stab(\sigma))\}$. The $d^1$-differentials in this chain complex can be
described explicitly as follows. For a simplex $\sigma \in \Sigma_p$, the restriction of $d^1$ to the summand $H_q(\text{stab}(\sigma))$ will be the alternating sum of partial boundary maps $d^1_q : H_q(\text{stab}(\sigma)) \to H_q(\text{stab}(\tau))$ where $\tau \in \Sigma_{p-1}$ is the orbit representative of the $i$th face $\partial_i \sigma$ and $d^1_q$ is induced by the inclusion $\text{stab}(\sigma) \to \text{stab}(\partial_i \sigma)$ followed by the conjugation that takes this stabilizer to $\text{stab}(\tau)$.

1.2. Proof of homology stability. Homology stability is proved by induction on the homology dimension $i$, starting with the trivial case $i = 0$. The sort of result we seek is that the stabilization $H_i(G_{n-1}) \to H_i(G_n)$ is an isomorphism for $n > \varphi(i)$ and a surjection for $n = \varphi(i)$, for a linear function $\varphi$ of positive slope.

The map $d = d^1 : E^1_{0,i} \to E^1_{-1,i}$ in the second spectral sequence constructed above is the map on homology induced by the inclusion of a vertex stabilizer into the whole group; by assumption this is the map $H_i(G_{n-1}) \to H_i(G_n)$ induced by the standard inclusion $G_{n-1} \to G_n$, so this is the map we are trying to prove is an isomorphism. In case the action of $G$ on $X$ is transitive on simplices of all dimensions, not just vertices, the $E^1$ page of the spectral sequence has the following form:

$$
\begin{array}{c|cccc}
  \text{Column } p & 0 & 1 & 2 & \cdots \\
  q = 0 & H_0(G_n) & \leftarrow & H_0(G_{n-1}) & \leftarrow & \cdots \\
  i-2 & \cdots & \leftarrow & H_{i-2}(G_{n-3}) & \leftarrow & \cdots \\
  i-1 & \cdots & \leftarrow & H_{i-1}(G_{n-2}) & \leftarrow & H_{i-1}(G_{n-3}) & \leftarrow & \cdots \\
  i & H_i(G_n) & \leftarrow & H_i(G_{n-1}) & \leftarrow & H_i(G_{n-2}) & \leftarrow & \cdots \\
\end{array}
$$

In the general case the $p$th column for $p \geq 2$ is modified by replacing the group $H_q(G_{n-p})$ by a direct sum of groups $H_q(G_{n-r})$ for values of $r$ in the range $1 \leq r \leq p$.

We first consider the argument for showing that $d$ is surjective. If $i - 1 \leq c(X)$, the connectivity of $X$, then the terms $E_{-1,i}^\infty$ must be zero for $p + q \leq i - 1$, and in particular $E_{-1,i}^\infty$ must be zero. We will show that every differential $d^r$ with target $E_{-1,i}^r$ for $r > 1$ is the zero map because its domain is the zero group, so the only differential that can do the job of killing $E_{-1,i}^r$ is $d$, which must therefore be onto. Thus it will suffice to show that $E_{-1,i}^2 = 0$ for $p + q \leq i$ and $q < i$. These groups are the reduced homology groups of $X/G$ with coefficients $\{H_q(\text{stab}(\sigma))\}$. We will argue that these coefficient groups can be replaced by $H_q(G)$, with a suitable induction hypothesis. Then if we assume $c(X/G) \geq i$, all the groups $E_{p,q}^2$ with $p + q \leq i$ and $q < i$ will be zero.
As explained earlier, the $d^1$ differentials are built from maps induced by inclusion followed by conjugation. These maps fit into commutative diagrams

$$
\begin{array}{ccc}
H_q(\text{stab}(\sigma)) & \longrightarrow & H_q(\text{stab}(\tau)) \\
\downarrow & & \downarrow \\
H_q(G) & \longrightarrow & H_q(G)
\end{array}
$$

where the vertical maps are induced by inclusion and the lower map is induced by conjugation in $G$, hence is the identity. If the vertical maps are isomorphisms we can then replace the coefficient groups in the $q$th row of the $E^1$ page by the constant groups $H_q(G)$. For example, in the special case displayed earlier, we would like the group $\tilde H_{i-1}(G_{n-3})$ and the groups to the left of it to be in the stable range, isomorphic to $H_{i-1}(G)$. Actually we can get by with slightly less, just having $H_{i-1}(G_{n-2})$ and the terms to the left of it isomorphic to the stable group and having $H_{i-1}(G_{n-3})$ mapping onto the stable group, since this is enough to guarantee that the homology of the chain complex at $H_{i-1}(G_{n-2})$ will be zero. Thus we want the relation $\varphi(i) \geq \varphi(i-1) + 2$. The corresponding relation for smaller values of $i$ will take care of lower rows, by the same argument. In the general case there will be more terms in $E^1_{p,q}$ besides the one $H_q(G_{n-p-1})$ but they will have $G_{n-p-1}$ replaced by $G_{n-r}$ with $n-r \geq n-p-1$ so they will still be in the stable range.

To summarize, we have shown that the stabilization $H_i(G_{n-1}) \rightarrow H_i(G_n)$ will be surjective if $\varphi(i) \geq \varphi(i-1) + 2$ and if $i-1 \leq c(X_n)$ and $i \leq c(X_n/G_n)$.

To prove that $d$ is injective the argument is similar, but with one extra step. If $i \leq c(X_n)$ the term $E^\infty_{0,i}$ will be zero, and then it will suffice to show that all differentials with target $E^1_{0,i}$ are zero, so the only way for $E^1_{0,i}$ to die is if $d$ is injective. We can argue that the terms $E^2_{p,q}$ are zero for $p+q \leq i+1$ and $q < i$ just as before, assuming again that $\varphi(i) \geq \varphi(i-1) + 2$ but with the inequality $i \leq c(X_n/G_n)$ replaced by $i+1 \leq c(X_n/G_n)$ since we have shifted one unit to the right in the spectral sequence. The extra step we need for injectivity of $d$ is showing that the differential $d^1: E^1_{1,i} \rightarrow E^1_{0,i}$ is zero. This follows from the assumption that for each edge $e$ of $X_n$ there is an element $g$ of $G_n$ taking one of the endpoints $v$ of $e$ to the other endpoint $w$, such that $g$ commutes with $\text{stab}(e)$. This guarantees that $d^1$ vanishes on the summand of $E^1_{1,i}$ corresponding to $e$ by our earlier description of $d^1$.

1.3. Examples.

**Example 1.1.** Suppose $X_n$ is contractible and $X_n/G_n$ is $(n-2)$-connected. The inequalities for $c(X_n)$ are then automatic, while for $c(X_n/G_n)$ the conditions are $i \leq n-2$ for surjectivity and $i+1 \leq n-2$ for injectivity. This is equivalent to saying $n \geq i+2$ for surjectivity and $n > i+2$ for injectivity. For the stable range function $\varphi(i) = 2i + c$ we therefore want to have $2i + c \geq i+2$ for all $i \geq 1$. This will hold if it holds for $i = 1$, so the optimal value of $c$ is $c = 1$. The conclusion is that $H_i(G_{n-1}) \rightarrow H_i(G_n)$ is an isomorphism for $n > 2i + 1$ and a surjection for $n = 2i + 1$. Note that the condition that $X_n$ be contractible can be weakened to its being just $(n-3)$-connected without affecting the resulting stable range.
Example 1.2. Modifying the preceding example, suppose $X_n$ is $(n-3)/2$-connected instead of being contractible, while $X_n/G_n$ is again $(n-2)$-connected. Then for surjectivity we have the added condition $i-1 \leq (n-3)/2$, or $n \geq 2i+1$, and for injectivity we have $i \leq (n-3)/2$ or $n \geq 2i+3$. Thus the optimal $\varphi$ is $\varphi(i) = 2i+2$, and the stabilization $H_i(G_{n-1}) \rightarrow H_i(G_n)$ is an isomorphism for $n > 2i+2$ and a surjection for $n = 2i+2$.

2. Connectivity tools

All of the complexes we will consider are complexes of the following type, which we shall call geometric. A geometric complex $X$ is a simplicial complex whose vertices correspond to isotopy classes of some type of nontrivial geometric object (arcs in a surface, curves in a surface, spheres in a 3-manifold, etc.), where trivial has different meanings in different contexts. A set of vertices $v_0, \ldots, v_k$ spans a $k$-simplex if representatives for the vertices can be chosen which are pairwise disjoint. The corresponding set of isotopy classes is also called a system, and the set of systems forms a partially ordered set (poset), whose geometric realization is the barycentric subdivision of the complex, denoted $\hat{X}$.

In this section we lay out a few general tools we will use for proving that various geometric complexes are highly connected.

2.1. Link arguments: rerouting disks to avoid bad simplices. We first describe a combinatorial method for altering a map from a triangulated manifold to a simplicial complex in order to push the image into a preferred subcomplex. Alterations are done locally, in the open star of one simplex at a time. This method of improving the map is called a link argument.

More precisely, let $M$ be a manifold with a finite triangulation, $X$ a simplicial complex, $f: M \rightarrow X$ a simplicial map and $Y$ a subcomplex of $X$. We want to homotope $f$ to a new map whose image lies in $Y$, and we want the homotopy to be constant on simplices whose images already lie in $Y$. We first identify a set of simplices in $X - Y$ as bad simplices, satisfying

1. any simplex with no bad faces is in $Y$, and
2. if two faces of a simplex are both bad, then their join is also bad.

We call simplices with no bad faces good simplices. Bad simplices may have good faces, or faces which are neither good nor bad. If $\sigma$ is a bad simplex we say a simplex $\tau \subset lk(\sigma)$ is good for $\sigma$ if any bad face of $\tau \ast \sigma$ is contained in $\sigma$. The simplices which are good for $\sigma$ form a subcomplex of $lk(\sigma)$ which we denote by $G_\sigma$.

Proposition 2.1. Let $f: M \rightarrow X, Y, and G_\sigma$ be as above. If $G_\sigma$ is $(\dim(M) - \dim(\sigma) - 1)$-connected for all bad simplices $\sigma$, then $f$ is homotopic to a map with image in $Y$, by a homotopy that is constant on simplices whose images already lie in $Y$.

Proof. Let $\mu$ be a maximal simplex of $M$ such that $\sigma = f(\mu)$ is bad. Then $f(lk(\mu)) \subset lk(\sigma)$ is actually contained in $G_\sigma$, since otherwise there is some $\nu \in lk(\mu)$ and face $\sigma_0$ of $\sigma$ such that $f(\nu) \ast \sigma_0$ is bad, in which case by property (2) $(f(\nu) \ast \sigma_0) \ast \sigma = f(\nu) \ast \sigma = f(\nu \ast \mu)$ is bad, contradicting maximality of $\mu$. 
Since $M$ is a manifold, $lk(\mu)$ is either contractible (if $\mu \subset \partial M$) or homeomorphic to $S^{n-k-1}$, where $n = \dim(M)$ and $k = \dim(\mu) \geq \dim(\sigma)$. Since $G_{\sigma}$ is $(n-k-1)$-connected, the restriction of $f$ to $lk(\mu)$ can be extended to a map $g: D^{n-k} \to G_{\sigma}$, which we may take to be simplicial. We retriangulate $st(\mu)$ as $\partial \mu \ast D^{n-k}$ and redefine $f$ on this new triangulation to be $f|_{\partial \mu \ast g}$ (see Figure 2).

The new map is homotopic to the old map, and agrees with the old map on all simplices of $Y$. Since simplices in $G_{\sigma}$ are good for $\sigma$, no interior simplices of this new triangulation have bad images. Since the triangulation of $M$ is finite, in this way we can eventually eliminate all bad simplices from the image, so that by property (1) the image lies in $Y$. □

We give two applications of this proposition which we will use in the rest of the paper.

**Corollary 2.2.** Let $Y$ be a subcomplex of an $n$-connected complex $X$, and suppose $X - Y$ has a set of bad simplices satisfying (1) and (2) above. If $G_{\sigma}$ is $(n - \dim(\sigma))$-connected for all bad simplices $\sigma$, then $Y$ is $n$-connected.

**Proof.** Let $f_0: S^i \to Y$ be a simplicial map with $i \leq n$. Since $X$ is $n$-connected, we can extend this to a map $f: D^{i+1} \to X$. Then Lemma 2 applies with $M = D^{i+1}$ to show $f$ is homotopic to a map $D^{i+1} \to Y$ which agrees with $f_0$ on $S^i$. □

Given any simplicial complex $X$ and a set of labels $S$, we can form a new simplicial complex $X^S$ whose simplices consist of the simplices of $X$ with vertices labeled by elements of $S$. Thus there are $|S|^{k+1}$ $k$-simplices of $X^S$ for each $k$-simplex of $X$.

**Corollary 2.3.** Let $X$ be a simplicial complex and $S$ a finite set of labels. If $X$ is $n$-connected and the link of each $k$-simplex in $X$ is $(n-k-1)$-connected, then $X^S$ is $n$-connected.

**Proof.** Let $f: S^i \to X^S$ be a simplicial map with $i \leq n$. Choose $s_0 \in S$, and let $Y$ be the subcomplex of $X^S$ consisting of simplices with all labels equal to $s_0$. Note that $Y$ is
isomorphic to $X$, which is $n$-connected; our goal is to homotope $f$ to have image in $Y$, where we can extend it over $D^{i+1}$.

Call a simplex bad if none of its vertex labels is equal to $s_0$. It is immediate that the set of bad simplices satisfies (1) and (2). If $\sigma$ is a bad simplex, then a simplex in $lk(\sigma)$ is good for $\sigma$ if and only if all of its labels are $s_0$, so that $G_\sigma$ is isomorphic to the link of the corresponding unlabeled simplex of $X$ and Proposition 2.1 applies.

Example 2.4. If $X$ is the $p$-simplex $\Delta^p$, one might think the lemma could be applied for all $n$ to conclude that $(\Delta^p)^S$ was contractible. However, it can only be applied for $n \leq p-1$ since for $n = p$ the link of the whole simplex would have to be $(-1)$-connected, i.e., nonempty, which is not the case. In fact $(\Delta^p)^S$ is the join of $p+1$ copies of the discrete set $S$, so it is $p$-dimensional and exactly $(p-1)$-connected if $S$ has more than one element.

2.2. Homotopy equivalence of posets. The geometric realization of a poset $P$ is the simplicial complex with one $k$-simplex for each totally ordered chain $p_0 < \cdots < p_k$ of $k+1$ elements $p_i \in P$. An order-preserving map (poset map) between posets induces a simplicial map of their geometric realizations. When we attribute some topological property to a poset or poset map we mean that the corresponding space or simplicial map has that property.

For a poset map $\phi: P \to Q$ the fiber $\phi_{\leq q}$ over an element $q \in Q$ is defined to be the subposet of $P$ consisting of all $p \in P$ with $\phi(p) \leq q$. The fiber $\phi_{\geq q}$ is defined analogously. The following statement is known as Quillen’s Fiber Lemma and is a special case of his Theorem A [19]. We supply an elementary proof.

Proposition 2.5. A poset map $\phi: P \to Q$ is a homotopy equivalence if all fibers $\phi_{\leq q}$ are contractible, or if all fibers $\phi_{\geq q}$ are contractible.

Proof. There is no difference between the two cases, so let us assume the fibers $\phi_{\geq q}$ are contractible. We construct a map $\psi: Q \to P$ inductively over the skeleta of $Q$ as follows. For a vertex $q_0$ we let $g(q_0)$ be any vertex in $\phi_{\geq q_0}$, which is non-empty since it is contractible. For an edge $q_0 < q_1$ both $g(q_0)$ and $g(q_1)$ then lie in $\phi_{\geq q_0}$ and we let $\psi$ map this edge to any path in $\phi_{\geq q_0}$ from $\psi(q_0)$ to $\psi(q_1)$. Extending $\psi$ over higher simplices $q_0 < \cdots < q_k$ is done similarly, mapping them to $\phi_{\geq q_0}$ extending the previously constructed map on the boundary of the simplex.

We claim that $\psi$ is a homotopy inverse to $\phi$. The composition $\phi \psi$ sends each simplex $q_0 < \cdots < q_k$ to the subcomplex $Q_{\geq q_0}$. These subcomplexes are contractible, having minimum elements, so one can construct a homotopy from $\phi \psi$ to the identity inductively over skeleta of $Q$. Similarly $\psi \phi$ is homotopic to the identity since it sends each simplex $p_0 < \cdots < p_k$ to the contractible subcomplex $\phi_{\geq \phi(p_0)}$.

We will often apply this proposition to the poset of simplices in some simplicial complex. The geometric realization of this poset is the barycentric subdivision of the complex, and the induced map is homeomorphic to the original simplicial map. The following lemma characterizes the poset fibers.
Lemma 2.6. Let $f: X \to Y$ be a simplicial map of simplicial complexes, $\hat{X}$ the poset of simplices in $X$, $\hat{Y}$ the poset of simplices in $Y$, and $\hat{f}: \hat{X} \to \hat{Y}$ the induced poset map. Then for any (closed) simplex $\sigma$ of $Y$ we have the following relationships:

(i) $\hat{f}_{\leq \sigma}$ is homeomorphic to $f^{-1}(\sigma)$.
(ii) $\hat{f}_{\geq \sigma}$ is homotopy equivalent to $\hat{f}^{-1}(\sigma)$.
(iii) $\hat{f}^{-1}(\sigma)$ is homeomorphic to $f^{-1}(y)$ where $y$ is the barycenter of $\sigma$.

Proof. Statement (i) is immediate from the definitions: $\hat{f}_{\leq \sigma}$ is the set of all simplices $\tau$ such that $f(\tau)$ is a face of $\sigma$.

On the other hand, $\hat{f}_{\geq \sigma}$ is the set of all simplices $\tau$ such that $f(\tau)$ has $\sigma$ as a face. Since $f$ is a simplicial map some face of $\tau$ maps to $\sigma$; let $\tau_{\sigma}$ be the (unique) maximal such face. The map $\tau \mapsto \tau_{\sigma}$ is a poset map $\hat{f}_{\geq \sigma} \to \hat{f}^{-1}(\sigma)$ whose upper fibers are contractible, having unique minimal elements. Thus $\hat{f}_{\geq \sigma}$ is homotopy equivalent to $\hat{f}^{-1}(\sigma)$, giving statement (ii).

Part (iii) is clear from the definitions. □

The following is an immediate consequence of Proposition 2.5 and Lemma 2.6:

Corollary 2.7. Let $f: X \to Y$ be a simplicial map of simplicial complexes. If $f^{-1}(\sigma)$ is contractible for all closed simplices $\sigma$ or if $f^{-1}(y)$ is contractible for all barycenters $y$, then $f$ is a homotopy equivalence.

2.3. Fiber connectivity.

Lemma 2.8. Let $f: X \to Y$ be a simplicial map of simplicial complexes. Suppose that $Y$ is $n$-connected and the fibers $f^{-1}(y)$ over the barycenters $y$ of all $k$-simplices in $Y$ are $(n-k)$-connected. Then $X$ is $n$-connected.

Proof. Given a map $g: S^i \to X$ which we can assume is simplicial, we want to extend this to a map $G: D^{i+1} \to X$ if $i \leq n$. In order to do this, we first consider the composition $h = fg: S^i \to Y$. Since $Y$ is $n$-connected, we can extend $h$ to a simplicial map $H: D^{i+1} \to Y$. We will use $H$ to construct $G$, which we will do inductively on the skeleta of the barycentric subdivision $D'$ of $D^{i+1}$.

We begin by replacing all complexes and maps by the associated posets of simplices and poset maps:

\[
\begin{array}{ccc}
\hat{S}^i & \overset{\hat{g}}{\longrightarrow} & \hat{X} \\
\downarrow & & \downarrow \hat{f} \\
\hat{D}^{i+1} & \overset{\hat{H}}{\longrightarrow} & \hat{Y}
\end{array}
\]

Let $\tau$ be a vertex of $D'$, so $\tau$ can be viewed as a simplex of $D^{i+1}$ or an element of $\hat{D}^{i+1}$. Since $H$ is simplicial, $\sigma = H(\tau)$ has dimension at most $i+1 \leq n+1$ in $Y$. By the hypothesis and Lemma 2.6, $\hat{f}_{\geq \sigma}$ is at least $(-1)$-connected, i.e., it is non-empty, so choose $x \in \hat{f}_{\geq \sigma}$ and set $G(\tau) = x$. We can assume this agrees with the given $g$ for $\tau \in \partial D'$. 
Now assume we have defined $G$ on the $(k-1)$-skeleton of $D'$, and let $\tau_0 < \cdots < \tau_k$ be a $k$-simplex of $D'$. Let $\sigma_i = H(\tau_i)$, and note that $\tilde{f}_{\geq \sigma_j} \circ \tilde{f}_{\geq \sigma_0}$ for all $j$. By construction, then, $G$ maps the boundary of the simplex to $\tilde{f}_{\geq \sigma_0}$. Since $f$ is a simplicial map, it can only decrease the dimension of a simplex, so $\dim(\sigma_0) \leq i + 1 - k \leq n + 1 - k$, and consequently $\tilde{f}_{\geq \sigma_0}$ is at least $(k-1)$-connected. Therefore we can extend $G$ over the interior of the $k$-simplex $\tau_0 < \cdots < \tau_k$, agreeing with the given $G$ on $\partial D'$. This gives the induction step in the construction of $G$. \hfill \Box

### 2.4. Flowing into a subcomplex

In this section we abstract the essential features of a surgery technique from [12] for showing that certain complexes of arcs on a surface are contractible, in order to more conveniently apply the method to several different situations later in the paper.

Let $Y$ be a subcomplex of a simplicial complex $X$. If $F: X \times I \to X$ is a deformation retraction into $Y$ then each $x \in X$ gives a path $F(x, t), 0 \leq t \leq 1$ starting at $x$ and ending in $Y$. In nice cases these paths fit together to give a flow on the complement of $Y$. What we want to do is to work backwards, constructing a deformation retraction by first constructing a set of flow lines. Our flow lines will intersect each open simplex of $X$ which is not contained in $Y$ either transversely or in a family of parallel line segments. To specify these line segments, for each simplex $\sigma \in X - Y$ we choose a preferred vertex $v = v_\sigma$ and a simplex $\Delta v$ in the link of $v$ in $X$ such that $\sigma \ast \Delta v$ is a simplex of $X$; then $\sigma \ast \Delta v$ is foliated by line segments parallel to the line from $v_\sigma$ to the barycenter of $\Delta v_\sigma$ (see Figure 3). To show that the flow ends up in $Y$ we measure progress by means of a complexity function assigning a non-negative integer to each vertex of $X$ and taking strictly positive values on vertices not in $Y$. This can be extended to be defined for all simplices of $X$, where the complexity of a simplex is the sum of the complexities of its vertices.

**Lemma 2.9.** Let $Y$ be a subcomplex of a simplicial complex $X$ with a complexity function $c$ as above. Suppose that for each vertex $v \in X - Y$ we have a rule for associating a simplex $\Delta v$ in the link of $v$ in $X$, and for each simplex of $X$ not contained in $Y$ we have a rule for picking one of its vertices $v_\sigma \in X - Y$ so that

(i) the join $\sigma \ast \Delta v_\sigma$ is a simplex of $X$. 

---

**Figure 3.** A simplex $\sigma$, its preferred vertex $v_\sigma$, and flow lines in $\Delta v_\sigma \ast \sigma$.
(ii) \( c(\Delta v) < c(v) \).

(iii) if \( \tau \) is a face of \( \sigma \) which contains \( v_\sigma \), then \( v_\tau = v_\sigma \).

Then \( Y \) is a deformation retract of \( X \).

**Proof.** For each simplex \( \sigma \) not contained in \( Y \) we construct flow lines in the simplex \( \sigma \ast \Delta v_\sigma \) as described above, starting at \( x \in \sigma \) and running parallel to the line from \( v_\sigma \) to the barycenter of \( \Delta v_\sigma \). In terms of barycentric coordinates in the simplex \( \sigma \ast \Delta v_\sigma \), viewed as weights on its vertices, we are shifting the weight on \( v_\sigma \) to equally distributed weights on the vertices of \( \Delta v_\sigma \), keeping the weights of other vertices fixed. When we follow the resulting flow on \( \sigma \ast \Delta v_\sigma \), all points that actually move end up with smaller complexity by condition (ii). Thus after a finite number of such flows across simplices, each point of \( \sigma \) follows a polygonal path ending in \( Y \). Condition (iii) guarantees that the resulting flow is continuous on \( X \), where we fix a standard Euclidean metric on each simplex and let each point flow at constant speed so as to reach \( Y \) at time 1. \( \square \)

**Surgery flows.** In this paper we will use Lemma 2.9 on various complexes of arcs and curves on surfaces. The complexity function will count the number of nontrivial intersection points with a fixed arc, curve, or set of curves, and the simplex \( \Delta v \) will be obtained using the surgery technique from [12] to decrease the number of intersection points. The vertex \( v_\sigma \) will be an “innermost” or “outermost” arc or curve of \( \sigma \), depending on the situation. In order for this surgery process to be well-defined we must first put each arc or curve system \( \{t_0, \ldots, t_k\} \) into normal form with respect to some fixed arc, curve, or curve system \( t \), so that each \( t_i \) has minimal possible intersection with \( t \) in its isotopy class. In all cases we consider these normal forms are easily shown to exist; furthermore they are unique up to isotopy through normal forms, apart from the special situation that one \( t_i \) is isotopic to \( t \), in which case this \( t_i \) can be isotoped across \( t \) from one side to the other without always being in normal form during the isotopy.

3. A simple example: the braid group

As a warm up for our main case of mapping class groups let us first show how the machinery gives a simple proof of homology stability for the classical braid groups \( B_n \), where we are viewing \( B_n \) as the mapping class group of an \( n \)-punctured disk.

We start by constructing a suitable “tethered” complex with a \( B_n \)-action. In fact, in this case the tethers will be all there is to the complex. Consider a fixed disk \( D \) with \( d \) distinguished points \( b_1, \ldots, b_d \) on the boundary and \( n \) marked points or punctures \( p_1, \ldots, p_n \) in the interior. A **tether** is an arc in \( D \) connecting some \( p_i \) to some \( b_j \) and disjoint from the other \( p_k \)’s and \( b_k \)’s. A **system of tethers** is a collection of tethers which are disjoint except at their endpoints, and with no two of the tethers isotopic. See Figure 4 for an example.

Let \( X = X_{n,d} \) be the simplicial complex having one \( k \)-simplex for each isotopy class of systems of \( k + 1 \) tethers, where the face relation is given by omitting tethers.

**Theorem 3.1.** \( X \) is contractible.

**Proof.** Choosing a single fixed tether \( t \), we will use a surgery flow to deform \( X \) into the star of \( t \). The flow will decrease the complexity of a system \( \sigma \) (in normal form with respect to \( t \),
which is defined to be the total number of points in the intersection of the interiors of $\sigma$ and $t$.

If $s$ is a tether which intersects $t$ at an interior point, let $x$ be the intersection point which is closest along $t$ to the end $b_i$ of $t$. Perform surgery on $s$ by cutting it at $x$ and moving both new endpoints down to $b_i$ (see Figure 5). This creates two new arcs which can be isotoped to be disjoint from $s$ except at their endpoints. One of these arcs joins $b_i$ to a puncture, and one joins $b_i$ to some (possibly different) $b_j$. Define $\Delta s$ to be the arc connecting $b_i$ to a puncture. Note that $\Delta s$ has smaller complexity than $s$.

The conditions of Lemma 2.9 are now met, with the star of $t$ as the subcomplex $Y$, by defining $v_\sigma$ to be the tether in $\sigma$ containing the point of $\text{int}(\sigma) \cap \text{int}(t)$ closest to $b_i$ along $t$. Thus $X$ deformation retracts to the star of $t$, which is contractible, hence $X$ is contractible. □

A system of tethers $\tau = \{t_1, \cdots, t_k\}$ is coconnected if the complement $D - \tau$ is connected. Note that a system is coconnected if and only if each arc in the system ends at a different puncture. Let $Y = Y_{n,d}$ be the subcomplex of $X_{n,d}$ consisting of isotopy classes of coconnected tether systems.

**Theorem 3.2.** The complex $Y = Y_{n,1}$ is contractible.

**Proof.** We prove that $Y$ is contractible by induction on the number $n$ of punctures. If $n = 1$ then $Y$ is a single point. For the induction step we will use a link argument (Corollary 2.2),
so we need to specify which simplices are bad. We do this by defining a simplex of $X = X_{n,1}$ to be bad if each tethered puncture has at least two tethers. We check (1) every simplex in $X$ which is not in $Y$ has a bad face, and (2) if $\sigma$ and $\tau$ are two bad faces of a simplex of $X$, the join $\sigma \ast \tau$ is also bad.

If $\sigma$ is a bad simplex, we also need to identify the subcomplex $G_\sigma$ of $lk(\sigma)$ consisting of simplices which are good for $\sigma$. In our case $\tau \in lk(\sigma)$ is good for $\sigma$ if and only if $\tau$ consists of single tethers to punctures which are not used by $\sigma$. The subcomplex $G_\sigma$ decomposes as a join $G_\sigma = Y(P_1) \ast Y(P_2) \ast \cdots \ast Y(P_r)$, where $P_1, \ldots, P_k$ are the components of the space obtained by cutting $D$ open along $\sigma$ and each $Y(P_i)$ is either empty or isomorphic to $Y_{n_i,d_i}$ for some $n_i < n$. Since the tethers in $\sigma$ are not isotopic, an innermost piece $P_i$ must have at least one puncture and only one boundary point, so $Y(P_i) \cong Y_{n_i,1}$ is contractible by induction on $n$. Hence the entire join $G_\sigma$ is contractible so the hypotheses of Corollary 2.2 are satisfied and we conclude that $Y$ is contractible. 

Theorem 3.3. The stabilization $H_i(B_{n-1}) \to H_i(B_n)$ is an isomorphism for $n > 2i + 1$ and a surjection for $n = 2i + 1$.

Proof. We use the spectral sequence constructed in Section 1.1 for the action of $B_n$ on the contractible complex $Y = Y_{n,1}$. Recall that this action arises from regarding $B_n$ as the group of isotopy classes of diffeomorphisms of the disk that are the identity on the boundary and permute the punctures $p_i$. We verify conditions (1)–(4) stated at the beginning of Section 1.

1. The action has only one orbit of $k$-simplices for each $k$, in particular it is transitive on vertices.
2. To see that the stabilizer of a $k$-simplex fixes the simplex pointwise, note that a set of $k + 1$ tethers coming out of the basepoint in the boundary of the disk has a natural ordering determined by an orientation of the disk at the basepoint, and this ordering is preserved by any diffeomorphism of the disk that is the identity on the boundary. The stabilizer of a $k$-simplex is therefore isomorphic to $B_{n-k-1}$.
3. For an edge of $Y$ corresponding to a pair of tethers there is a diffeomorphism of the disk supported in a neighborhood of the two tethers that interchanges the punctures at the ends of the tethers and takes the first tether to the second or vice-versa. This diffeomorphism gives an element of $B_n$ commuting with the stabilizer of the edge.
4. $Y$ is contractible by Theorem 3.2. The quotient $Y/B_n$ can be identified with the quotient of the standard simplex $\Delta^{n-1}$ obtained by identifying all of its $k$-dimensional faces for each $k$, where the identification preserves the ordering of the vertices. Thus $Y/B_n$ is a $\Delta$-complex (or semisimplicial complex) with one $k$-simplex for each $k \leq n - 1$. It is easy to see that $Y/B_n$ is simply-connected. Its cellular chain complex has a copy of $Z$ in each dimension $k \leq n - 1$ with boundary maps that are alternately zero and isomorphisms. Therefore the reduced homology groups of $Y/B_n$ are trivial below dimension $n - 1$, while $H_{n-1}(Y/B_n)$ is trivial when $n$ is odd and $Z$ when $n$ is even. Thus $Y/B_n$ is $(n-2)$-connected.

Example 1.1 now yields the statement of the theorem. 

It is not hard to deduce more from the spectral sequence:
Theorem 3.4. When $n$ is odd the stabilization $H_i(B_{n-1}) \to H_i(B_n)$ is an isomorphism for all $i$. Also, $H_i(B_n) = 0$ for $i \geq n$ (for $n$ of either parity).

Proof. We look more closely at the spectral sequence used in the proof of Theorem 3.3 above. The observation we used there to verify condition (3) holds more generally to show that for any system $\sigma$ of $k + 1 \geq 2$ tethers there is a diffeomorphism of the disk permuting the punctures and supported in a neighborhood of the tethers that takes any subset of $k$ of the tethers to any other set of $k$ of the tethers, preserving their natural order and commuting with $\text{stab}(\sigma)$. This implies that each term of the map $d^1: E^1_{p,q} = \text{stab}(\sigma) \to E^1_{p-1,q}$ is the same, so for $p$ odd $d^1$ is zero, and for $p$ even $d^1$ is the map induced by inclusion.

Now assume $n$ is odd. Then by induction on $n$ the differential $d^1: E^1_{p,q} \to E^1_{p-1,q}$ is an isomorphism for $p$ even, $p > 0$. The spectral sequence has the following form:

\[
\begin{array}{c|cccc}
  i & H_i(B_n) & \leftarrow & H_i(B_{n-1}) & \leftarrow 0 & H_i(B_{n-2}) & \leftarrow & \cdots \\
  i-1 & \cdots & \leftarrow & H_{i-1}(B_{n-1}) & \leftarrow 0 & H_{i-1}(B_{n-2}) & \leftarrow & \cdots \\
  i-2 & \cdots & \leftarrow & H_{i-2}(B_{n-3}) & \leftarrow 0 & \cdots \\
  q = 0 & H_0(B_n) & \leftarrow & H_0(B_{n-1}) & \leftarrow 0 & H_0(B_{n-2}) & \leftarrow & \cdots \\
  p = -1 & 0 & 1 & 2 & \cdots \\
\end{array}
\]

The last nonvanishing column on the right is the $p = n - 1$ column since $Y_{n,1}$ has dimension $n - 1$. The differentials $d^1$ originating in this column are isomorphisms since $n$ is odd. (The only nonzero term in this column is $H_0(B_0) = \mathbb{Z}$ since $B_0$, like $B_1$, is the trivial group.) Thus in the $E^2$ page all the terms to the right of the $p = 0$ column vanish. Since the spectral sequence converges to 0 and no differentials beyond the $E^1$ page can be nonzero, it follows that the differentials $d^1: H_i(B_{n-1}) \to H_i(B_n)$ must be isomorphisms for all $i$.

For the second statement of the theorem we again look at the $E^1$ page of the spectral sequence. Along the $p + q = n - 1$ diagonal are the groups $H_j(B_j)$. By induction on $n$ all the terms above this diagonal are zero except possibly in the $p = -1$ column. Hence the terms $H_i(B_n)$ in this column must vanish for $i \geq n$ since the spectral sequence converges to zero. \hfill \Box

The fact that $H_i(B_n)$ vanishes for $i \geq n$ is also a consequence of the well-known fact that there is a $K(B_n,1)$ CW complex of dimension $n - 1$ (see, e.g., [8]). Arnold proved the statements in the two preceding theorems in [1] by methods not involving spectral sequences.

Arnold also computed the homology of the pure braid subgroup $P_n \subset B_n$ in [2] and it does not stabilize. What goes wrong with the proof when we restrict the action of $B_n$ on $Y$ to $P_n$?
is that we no longer have a transitive action on vertices of $Y$, a crucial requirement for the spectral sequence argument.

4. Curve complexes

For a compact orientable surface $S = S_{g,s}$ of genus $g$ with $s$ boundary components the classical curve complex $C(S)$ has as its vertices the isotopy classes of embedded curves (circles) in $S$ which are nontrivial, not bounding a disk and not isotopic to a component of $\partial S$. A set of vertices of $C(S)$ spans a simplex if the corresponding curves can be isotoped to be all disjoint, so they form a curve system. Let $C^0(S)$ be the subcomplex whose simplices are the isotopy classes of coconnected curve systems, systems with connected complement.

We will show the complexes $Ch(S)$ and $TCh(S)$ of chains and tethered chains are highly connected by comparing them with $C^0(S)$ via the natural forgetful maps

$$TCh(S) \to Ch(S) \to C^0(S)$$

where the first map forgets the tethers and the next map forgets the second curves in chains $(a,b)$. The first step is to show that $C^0(S)$ is highly connected, which we do in the present section. The results in this section are due originally to Harer in [10] and we follow the same overall strategy while simplifying the proofs of several of the individual steps.

The connectivity result for $C^0(S)$ will follow from a corresponding connectivity result for $C(S)$ via a link argument. To prove $C(S)$ is highly connected when $\partial S$ is nonempty the idea is to compare $C(S)$ with three other complexes in a sequence

$$A(S,\partial S) \supset A_\infty(S,\partial_0 S) \simeq S(S,\partial S) \simeq C(S)$$

The case that $S$ is closed will be deduced from the non-closed case.

We start by defining $A(S,\partial_0 S)$. An arc system on a bounded surface $S$ is a set of disjoint embedded arcs with endpoints on the boundary $\partial S$, such that no arc is isotopic to an arc in $\partial S$ and no two arcs in a system are isotopic to each other, where all isotopies of arcs are required to keep their endpoints in $\partial S$. Choosing a component $\partial_0 S$ of $\partial S$, the complex $A(S,\partial_0 S)$ has as its $k$-simplices the isotopy classes of systems of $k + 1$ arcs whose endpoints all lie in $\partial_0 S$.

Proposition 4.1. The complex $A(S,\partial_0 S)$ is contractible whenever it is nonempty, i.e., when $S$ is not a disk or annulus.

Proof. This is an application of Lemma 2.9, using surgery to flow into the star of a fixed “target” arc $a$. The complexity of a system that intersects $a$ minimally within its isotopy class is defined as the number of intersection points with $a$. To do the surgery we first choose an orientation for $a$. An arc $b$ crossing $a$ is cut into two arcs at the point where it meets $a$ nearest the head of $a$, and the two new endpoints are moved to the head of $a$ to produce a new arc system $\Delta b$ meeting $a$ in one fewer point than $b$. The function $\sigma \mapsto b_\sigma$ assigns to a system $\sigma$ the arc of $\sigma$ meeting $a$ at the point closest to the head of $a$. □

We define an arc system in $A(S,\partial_0 S)$ to be at infinity if it has some complementary component which is neither a disk nor an annular neighborhood of a boundary component. (The terminology comes from the fact, observed by Harer, that arc systems at infinity can be
identified with rational points in the boundary of the Teichmüller space of the surface.) Arc systems at infinity form a subcomplex $A_\infty(S, \partial_0 S)$. A calculation using Euler characteristics shows that it takes at least $(2g + s - 1)$ arcs to cut $S$ into disks and annuli when $g > 0$, so in this case $A_\infty(S, \partial_0 S)$ contains the entire $(2g + s - 3)$-skeleton of $A(S, \partial_0 S)$. When $g = 0$ $A_\infty(S, \partial_0 S)$ contains the $(s - 4)$-skeleton of $A(S, \partial_0 S)$. Hence:

**Corollary 4.2.** $A_\infty(S, \partial_0 S)$ is $(2g + s - 4)$-connected if $g > 0$ and $(s - 5)$-connected if $g = 0$.

Now define the *subsurface complex* $S(S, \partial_0 S)$ to be the poset of isotopy classes of compact connected subsurfaces $S'$ of $S$ such that $S'$ contains $\partial_0 S$ and $\partial S' - \partial S$ is a nonempty curve system in $S$, possibly containing parallel copies of the same curve. In particular, no component of $\partial S' - \partial_0 S$ bounds a disk in $S$ or is isotopic to a component of $\partial S$.

To each arc system $\alpha$ with $\partial \alpha \subset \partial_0 S$ we can associate a subsurface $S(\alpha)$ of $S$ by first taking a regular neighborhood $N$ of $\alpha \cup \partial_0 S$ and then adjoining any components of $S - N$ that are disks or annuli with one boundary circle contained in $\partial S$ (see Figure 6). Thus the simplices of $A_\infty(S, \partial_0 S)$ correspond to systems $\alpha$ for which $S(\alpha) \neq S$, and $\alpha \mapsto S(\alpha)$ is a map $f: \hat{A}_\infty(S, \partial_0 S) \to S(S, \partial_0 S)$ where the hat denotes the poset of simplices in $A_\infty(S, \partial_0 S)$. This map is a poset map since $\alpha \subset \beta$ implies $S(\alpha) \subset S(\beta)$.

**Proposition 4.3.** The map $f: \hat{A}_\infty(S, \partial_0 S) \to S(S, \partial_0 S)$ is a homotopy equivalence.

**Proof.** We apply Quillen’s fiber lemma, Proposition 2.5. If $S'$ is a subsurface of $S$ then $f_{\leq S'}$ is all arc systems $\alpha$ with $S(\alpha) \subset S'$, so this is $\hat{A}(S', \partial_0 S)$. Since $S'$ is not a disk or annulus, $\hat{A}(S', \partial_0 S)$ is contractible by Proposition 4.1. $\square$

Given a curve system $\gamma$, let the subsurface $S(\gamma) \subset S$ be the component of the complement of a regular neighborhood of $\gamma$ containing $\partial_0 S$ (see Figure 7). Note that if $\gamma \subset \gamma'$ then
Proposition 4.4. The map $g: \hat{C}(S) \to S(S, \partial_0 S)$ is a homotopy equivalence.

Proof. We again apply Proposition 2.5. For a subsurface $S'$ in $S(S, \partial_0 S)$, the fiber $g_{\geq S'}$ consists of curve systems in $S - S'$, where curves are allowed to be parallel to curves of the system $\gamma(S') = \partial S' - \partial S$. In particular, $\gamma(S')$ is in the fiber, and $\gamma(S')$ can be added to any curve system in the fiber, so the poset maps $\gamma \mapsto \gamma \cup \gamma(S') \mapsto \gamma(S')$ give a deformation retraction of $g_{\geq S'}$ to the point $\gamma(S')$.

Corollary 4.5. If $\partial S$ is not empty, then $C(S)$ is $(2g + s - 4)$-connected if $S$ has genus $g > 0$ and $(s - 5)$-connected if $S$ has genus $g = 0$.

Corollary 4.6. If $S$ has genus 0, then $C(S)$ is homotopy equivalent to a wedge of spheres of dimension $s - 4$.

Proof. If $S$ has genus 0, $C(S)$ is $(s - 5)$-connected by the preceding corollary, and it is $(s - 4)$-dimensional, so it is homotopy equivalent to a wedge of spheres of dimension $s - 4$.

Remark. In fact $C(S)$ is homotopy equivalent to a wedge of spheres in all cases, where the dimension of the spheres is one greater than the connectivity stated above. This was proved by Harer in [11]. However, this fact is not needed to prove homology stability.

4.1. Curves on closed surfaces. There is a map $\phi: C(S_{g,1}) \to C(S_{g,0})$ induced by filling in the boundary circle of $S_{g,1}$ with a disk. We remark that the dimension of $C(S_{g,1})$ is one more than that of $C(S_{g,0})$ when $g > 1$ since maximal curve systems cut $S$ into pairs of pants. For $g = 1$ the map $C(S_{1,1}) \to C(S_{1,0})$ is an isomorphism.

Theorem 4.7. The map $\phi: C(S_{g,1}) \to C(S_{g,0})$ is a homotopy equivalence for each $g \geq 1$.

Proof. We may assume $g > 1$. Choose a curve system $\delta$ which cuts $S_{g,1}$ into pairs of pants. One of these pairs of pants $P$ will contain the circle $\partial S_{g,1}$. Let $d_1$ and $d_2$ be the other two components of $\partial P$. We may assume that all curve systems $\gamma$ in $S_{g,1}$ are in normal form with respect to $\delta$, so $\gamma$ intersects $\delta$ transversely in the minimum number of points among all systems isotopic to $\gamma$. This minimality is equivalent to the “no bigon” condition that $S$ contains no disk whose boundary consists of an arc in $\gamma$ and an arc in $\delta$. If two systems in normal form with respect to $\delta$ are isotopic, then they are isotopic through systems transverse to $\delta$, except that curves in $\gamma$ isotopic to curves in $\delta$ can be pushed from one side of $\delta$ to the other and such an isotopy cannot be transverse to $\delta$ at all times.

If a curve system $\gamma$ is in normal form with respect to $\delta$ then each component arc of $\gamma \cap P$ either crosses $P$ from $d_1$ to $d_2$, or it enters $P$, goes around $\partial S_{g,1}$, and leaves by crossing the same $d_i$ that it crossed when it entered $P$. An arc of the latter type we call a return arc. Note that all return arcs of $\gamma$ must have their endpoints on the same $d_i$.

Step 1. We will use surgery to flow from $C(S_{g,1})$ to the subcomplex $C_0$ consisting of curve systems without return arcs. Let $c$ be a curve in normal form with respect to $\delta$ that contains return arcs. Let $b$ be the innermost of these return arcs, the one closest to $\partial S_{g,1}$. Pushing
Step 1. Let \( \phi : C_0 \to C(S_{g,0}) \) be the simplicial map obtained by filling in \( \partial S_{g,1} \) with a disk \( D \). We show that the induced poset map \( \hat{\phi} \) on simplices is a homotopy equivalence using Corollary 2.7 by showing that its fibers \( \hat{\phi}^{-1}(\sigma) \) are contractible. The map \( \phi \) is surjective, so choose a system \( \gamma \) in \( C_0 \) with \( \phi(\gamma) = \sigma \). We may assume \( \gamma \) is in normal form with respect to \( \delta \), and then \( \sigma \) will also be in normal form with respect to \( \phi(\delta) \) in \( S_{g,0} \) since \( \gamma \) has no return arcs.

If \( \gamma \) does not intersect \( P \), then \( \gamma \) is the unique element of \( \hat{\phi}^{-1}(\sigma) \). If \( \gamma \) intersects \( P \) in \( k \geq 1 \) arcs \( a_1, \ldots, a_k \), contained in curves \( c_1, \ldots, c_k \) (some of which may be the same), number the arcs \( a_i \) so that the filling disk \( D \) lies between \( a_1 \) and \( a_k \), and \( a_i \) is adjacent to \( a_{i+1} \) for \( i = 1, \ldots, k-1 \). If we push \( a_1 \) across \( D \) we convert \( c_1 \) to a new curve \( c_1' \) which can be isotoped to be disjoint from \( c_1 \), so that \( \gamma_1 = \gamma \cup c_1' \) is another simplex of \( C_0 \) (possibly equal to \( \gamma \)) with \( \phi(\gamma_1) = \sigma \). Let \( \gamma_0 = \gamma_1 - c_1' \) and \( \gamma_2 = \gamma_1 - c_1 \), so we have inclusions \( \gamma_0 \subset \gamma_1 \subset \gamma_2 \).

Having pushed \( a_1 \) across \( D \), we can then push \( a_2 \) across \( D \) producing new simplices \( \gamma_3 \) and \( \gamma_4 \) with \( \gamma_2 \subset \gamma_3 \subset \gamma_4 \), and then we can iterate for \( a_3, \ldots, a_k \). (See Figure 9.) Moreover, after \( a_k \) has been pushed across \( D \) we can repeat the whole process, pushing each of the new arcs \( a_1, \ldots, a_k \) across \( D \) in turn. In similar fashion, we could have started the process by pushing in the opposite direction, first pushing \( a_k \) across \( D \), then \( a_{k-1} \), and so on to obtain
\[ \gamma_0 \subset \gamma_1 \supset \gamma_2 \subset \gamma_3 \supset \gamma_4 \cdots. \] In this way we get all the simplices in \( \hat{\phi}^{-1}(\sigma) \), without any repetitions, and they form a bi-infinite chain
\[ \cdots \subset \gamma_2 \subset \gamma_1 \supset \gamma_0 \subset \gamma_1 \supset \gamma_2 \subset \gamma_3 \supset \gamma_4 \cdots \]
whose realization is a copy of the real line \( \mathbb{R} \) and so is contractible. \( \square \)

4.2. Coconnected curve systems.

**Theorem 4.8.** The complex \( C^0(S) \) of coconnected curve systems on a surface \( S \) of genus \( g \) is \( (g - 2) \)-connected.

**Proof.** This is a link argument, an application of Corollary 2.2 with \( X = C(S) \) and \( Y = C^0(S) \). We have shown above that \( C(S) \) is \( (2g - 3) \)-connected, so it is \( (g - 2) \)-connected if \( g > 0 \).

To each curve system we associate a dual graph, with a vertex for each complementary component of the system and an edge for each curve. Thus a curve system is coconnected if and only if its dual graph has one vertex and all edges are loops. A bad simplex in \( C(S) \) is a system of curves for which no edges of the dual graph are loops. This is equivalent to saying that each curve in the system separates the complement of the other curves. Conditions (1) and (2) in Section 2.1 are satisfied for this notion of badness. For a bad simplex \( \sigma \), the complex \( G_{\sigma} \) is the join of the complexes \( C^0(S_i) \) for the components \( S_i \) of \( S - \sigma \). Either the genus \( g_i \) of \( S_i \) is smaller than \( g \) or \( S_i \) has fewer boundary components than \( S \) so we may proceed by induction on the lexicographically ordered pair \( (g, s) \). The fact that connectivity plus two is additive for joins implies that \( G_{\sigma} \) has connectivity \( \sum g_i - 2 \). Cutting \( S \) along the curves of \( \sigma \) can decrease the total genus by at most \( \dim(\sigma) \) since each cut decreases genus by at most one and cutting along the last curve cannot decrease genus since \( \sigma \) is bad. (This estimate is best possible in the case that the dual graph to \( \sigma \) consists of two vertices joined by a number of edges.) Thus the connectivity of \( G_{\sigma} \) is at least \( \sum g_i - 2 \geq g - \dim(\sigma) - 2 \) and Corollary 2.2 applies to show that \( C^0(S) \) is \( (g - 2) \)-connected. \( \square \)

**Remark 4.9.** There is an easy argument showing that \( H_{g-1}(C^0(S_{g,s})) \) is nonzero for all \( g \geq 1 \) and \( s \geq 0 \). Choosing \( g \) disjoint copies of \( S_{1,1} \) in \( S_{g,s} \) gives an embedding of the join of \( g \) copies of \( C^0(S_{1,1}) \) into \( C^0(S_{g,s}) \) as a subcomplex. The complex \( C^0(S_{1,1}) = C(S_{1,1}) \) is an infinite discrete set, so the join is homotopy equivalent to the wedge of an infinite number of copies of \( S^{g-1} \). The inclusion map of the join into \( C^0(S_{g,s}) \) induces an injection on \( H_{g-1} \) since both complexes have dimension \( g - 1 \) so no nontrivial \( (g - 1) \)-dimensional cycle in the join can bound in \( C^0(S_{g,s}) \). Thus \( H_{g-1}(C^0(S_{g,s})) \) is nontrivial, and in fact is free abelian of infinite rank.

5. Complexes of connecting arcs

For a given surface \( S = S_{g,s} \) with \( s > 0 \), let \( P \) and \( Q \) be disjoint nonempty compact one-dimensional submanifolds of \( \partial S \). We will be interested in nontrivial arcs in \( S \) having one endpoint in \( P \) and the other in \( Q \), where an arc is considered trivial if it is isotopic to an arc in \( \partial S \) that meets \( P \cup Q \) only at its endpoints. Here we allow isotopies of arcs which move their endpoints, provided that the endpoints remain in \( P \) or \( Q \) at all times. The isotopy
classes of such nontrivial arcs form the vertices of a simplicial complex $A(P, Q)$, where as usual a set of $k + 1$ vertices spans a $k$-simplex if they correspond to a system of disjoint arcs.

To deduce the high connectivity of $\text{Ch}(S)$ from that of $C_0(S)$ we will need to know that $A(P, Q)$ is highly connected. Unfortunately both the statement of the connectivity bound and its proof are rather technical. Eventually all that will be needed are special cases whose statements are less complicated, but the proof seems to require the full generality.

The connectivity bound will depend on several parameters. First there are the genus $g$ of $S$ and the number $s$ of boundary components, but the number $s'$ of boundary components that meet $P$ or $Q$ will also play a role. In addition there are two other relevant parameters $t$ and $u$ which we now define. We can label the arc components of $\partial S - (P \cup Q)$ as mixed or pure according to whether they have one endpoint in $P$ and one in $Q$ or not. The number of mixed arcs of $\partial S$ is an even number $2t$ and the number of pure arcs is $u$. Note that when $u = 0$, so the arc components of $P$ and $Q$ alternate in $\partial S$, the number $t$ is the number of arc components of $P$ (or $Q$).

**Theorem 5.1.** The complex $A(P, Q)$ is $(4g + s + s' + t + u - 6)$-connected.

In the case that $P$ and $Q$ have no circle components the high connectivity of $A(P, Q)$ was stated by Harer in [10] although the proof given there was incomplete. The first full proof is due to Wahl in [25] and the key ideas we use here are from that proof. Another exposition can be found in [26].

If $P$ or $Q$ does contain a circle component then $A(P, Q)$ is easily seen to be contractible by a surgery argument; this is a special case of the following lemma.

**Lemma 5.2.** If some component $\partial_0 S$ of $\partial S$ contains one of $P$ and $Q$ but not both, then $A(P, Q)$ is contractible.

**Proof.** Fix a target arc in $A(P, Q)$ having one end in $\partial_0 S$, and do surgery toward this end. Each surgery cuts an arc from $P$ to $Q$ into two arcs, one of which goes from $P$ to $Q$, and this arc is retained while the other one is discarded. The new arc from $P$ to $Q$ is nontrivial since it connects different components of $\partial S$.

Lemma 5.2 is the first step in reducing high connectivity of $A(P, Q)$ to the case when the arcs of $P$ and $Q$ alternate in $\partial S$. The following lemmas complete this reduction.

**Lemma 5.3.** If some component $\partial_0 S$ of $\partial S$ contains three consecutive arc components of $P$ then $A(P, Q)$ is contractible.

**Proof.** If $\partial_0 S$ contains no components of $Q$ then Lemma 5.2 applies. Therefore, labeling the consecutive components $P_1, P_2, P_3$ in order, we may assume there is a component $Q_1$ of $Q$ adjacent to $P_1$. Let $a$ be a boundary-parallel arc from $Q_1$ to $P_2$ and use surgery to flow into the star of $a$ (see Figure 10). Note that the only arcs which are not already in the star of $a$ are those landing in $P_1$, so the surgery process can be thought of as shifting these to land in $P_3$ instead. The presence of the arc $P_3$ guarantees that the shifted arc is nontrivial if the original arc was nontrivial.
Despite the above lemmas, the complex $A(P, Q)$ is not always contractible. For example, when $S$ is a pair of pants and $P$ and $Q$ are single arcs in the same component of $\partial S$ then $A(P, Q)$ is zero-dimensional with an infinite number of vertices. The following lemma shows that connectivity increases when some component of $P$ is separated into two pieces.

**Lemma 5.4.** Suppose some component of $\partial S$ contains two consecutive arc components $P_1$ and $P_2$ of $P$ surrounded by components $Q_1$ and $Q_2$ of $Q$ (possibly $Q_1 = Q_2$). Let $P'$ be obtained from $P$ by deleting $P_2$ (or, equivalently, by merging $P_1$ and $P_2$). If $A(P', Q)$ is $k$-connected, then $A(P, Q)$ is $(k + 1)$-connected.

**Proof.** This situation is illustrated in Figure 11. Let $a$ be a boundary-parallel arc from $Q_1$ to $P_2$ and let $a'$ be a boundary-parallel arc from $P_1$ to $Q_2$. Let $A'$ be the subcomplex of $A(P, Q)$ consisting of arc systems not containing $a'$. Then $A'$ is contractible, using the shifting argument in Lemma 5.3 to deform it into the star of the arc $a$. The whole complex $A(P, Q)$ is the union of $A'$ and the star of $a'$, and the intersection of these two contractible subcomplexes is the link of $a'$. Hence by collapsing $A'$ to a point we see that $A(P, Q)$ is homotopy equivalent to the suspension of the link of $a'$. This link can be identified with $A(P', Q)$, since there are no arcs disjoint from $a'$ on the side containing $P_2$. \[\square\]

We next make a further reduction to the case that no component of $\partial S$ is disjoint from both $P$ and $Q$:

**Lemma 5.5.** Suppose $\partial S$ contains a component $\partial_0 S$ disjoint from $P$ and $Q$. Then the connectivity of $A(P, Q)$ is one greater than the connectivity of the complex $A(P, Q)$ for the surface obtained from $S$ by filling in $\partial_0 S$ with a disk, assuming that the complex $A(P, Q)$ for $S$ is nonempty.

**Proof.** Choose adjacent arcs $P_1$ and $Q_1$ in another component $\partial_1 S$ of $\partial S$, and consider arcs in $S$ from $P_1$ to $Q_1$ that are nontrivial in $S$ but become trivial in the surface $S'$ obtained by
filling in $\partial_0 S$ with a disk. Such an arc $a$ cuts off an annulus from $S$ with $\partial_0 S$ as one of its boundary components and with the other boundary component consisting of $a$, parts of $P_1$ and $Q_1$, and an arc $c$ of $\partial_1 S$ between $P_1$ and $Q_1$. (If there are two choices for an arc $c$ between $P_1$ and $Q_1$, we fix one choice throughout.) We call an arc $a$ satisfying these conditions a special arc. Up to isotopy, special arcs correspond bijectively with arcs $b$ in $S$ from $c$ to $\partial_0 S$. See Figure 12.

Note that special arcs exist since we assume $A(P, Q)$ is nonempty. Note also that there are no edges in $A(P, Q)$ joining vertices defined by special arcs since two special arcs giving distinct vertices of $A(P, Q)$ must intersect, otherwise one of the two would have to lie in the annulus cut off by the other, which would force them to be isotopic.

The subcomplex $A'$ of $A(P, Q)$ consisting of simplices having no vertices that are special arcs deformation retracts onto the link of any special arc $a$ by a flow in which arcs that meet the $b$ arc for $a$ are pushed across $\partial_0 S$ as in Figure 12. Since we are pushing nonspecial arcs, we never get trivial arcs and the flow is well-defined.

We can regard $A(P, Q)$ as being obtained from $A'$ by attaching the stars of all the special vertices along their links. The union of $A'$ with one of these stars is contractible by the preceding paragraph, and by collapsing this contractible subcomplex of $A(P, Q)$ to a point we see that $A(P, Q)$ is homotopy equivalent to a wedge of suspensions of the links of all the other special arcs. (In the special case that the links of special arcs are empty this conclusion still holds if we regard suspension as join with $S^0$.) Since we can identify the link of a special arc with the complex $A(P, Q)$ for $S'$, the result follows.

To prove Theorem 5.1 we will use a somewhat intricate link argument that involves embedding $A(P, Q)$ in a larger complex for which high connectivity is easier to prove. This larger complex is obtained by allowing arcs with both endpoints in $P$ or both endpoints in $Q$, as well as arcs with one endpoint in each. Thus the decomposition of $P \cup Q$ into $P$ and $Q$ no longer plays a role, so we can forget about $Q$ for the moment and consider the complex $A(P)$ of systems of nontrivial arcs with endpoints in the compact submanifold $P$ of $\partial S$, where trivial arcs are now arcs that are isotopic to arcs in $\partial S$ that either meet $P$ only in their endpoints or are completely contained in $P$. 

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure12.png}
\caption{Pushing arcs across $\partial_0 S$.}
\end{figure}
Lemma 5.6. $A(P)$ is contractible except in two cases: (a) If $S$ is a disk then $A(P)$ is homotopy equivalent to $S^{p-4}$ where $p$ is the number of arcs in $P$. (b) If $S$ is an annulus with $P$ contained in one of its boundary circles then $A(P)$ is homotopy equivalent to $S^{p-2}$.

In fact $A(P)$ is homeomorphic to a sphere in cases (a) and (b), although we will not need this refinement. See for example [12] for a proof.

Proof. If some boundary circle of $S$ contains exactly one component of $P$, we can fix a target arc $a$ with an endpoint in this component and use a surgery flow to deform all of $A(P)$ into the star of $a$, so that $A(P)$ is contractible in these cases. Surgery is done towards the chosen endpoint and trivial arcs are discarded. The fact that there is only one component of $P$ in this boundary circle guarantees that at least one of the arcs produced by each surgery is nontrivial.

The general case reduces to this special case by an argument similar to the one used in Lemma 5.4. The assertion is that adding an extra arc component to $P$ in a component of $\partial S$ that already contains at least one arc component of $P$ has the effect of suspending $A(P)$, up to homotopy equivalence. Referring to Figure 13, suppose $P$ is obtained from $P'$ by adding the arc $P_2$ next to an existing arc $P_1$. The arcs adjacent to $P_1$ and $P_2$ are labeled $P_3$ and $P_4$ as shown, where $P_3 = P_2$ if the total number of arcs in this component of $\partial S$ is three, or $P_3 = P_2$ and $P_4 = P_1$ if the total number of arcs is two. Let $A'$ be the subcomplex of $A(P)$ formed by arc systems not containing the arc $a'$. Then $A'$ deformation retracts into the star of $a$ by shifting arcs going to $P_1$ so that they go to $P_2$ instead. Thus $A'$ is contractible, and collapsing it to a point converts $A(P)$ into the suspension of the link of $a'$, which is a copy of $A(P')$.

If $S$ is not a disk or annulus with $P$ contained in one boundary circle, then this reduction produces a non-empty complex $A(P)$ for which the surgery argument applies, and we conclude that $A(P)$ is contractible in these cases as well.

If $S$ is a disk there must be at least four components in $P$ for $A(P)$ to be nonempty. If there are exactly four components then $A(P)$ consists of two points, so it is a 0-sphere. If $S$ is an annulus with $P$ contained in one boundary circle, then there must be at least two components in $P$ for $A(P)$ to be non-empty, and when there are two components $A(P)$ is again $S^0$. In either case, the reduction shows that adding a component to $P$ suspends $A(P)$, up to homotopy equivalence, giving statements (a) and (b).

Figure 13. The suspension argument again.
Figure 14. The cases for which $A(P, Q)$ is empty.

- $g$ is the genus of $S$
- $s$ is the number of boundary circles of $S$
- $s'$ is the number of boundary circles meeting $P \cup Q$
- $2t$ is the number of mixed arcs, i.e., arcs in $\partial S - (P \cup Q)$ joining a component of $P$ to a component of $Q$
- $u$ is the number of pure arcs, i.e., arcs in $\partial S - (P \cup Q)$ joining two components of $P$ or two components of $Q$.

and we want to show that $A(P, Q)$ is $(4g + s + s' + t + u - 6)$-connected.

Proof. Recall that we have assumed both $P$ and $Q$ are nonempty. We will also want to assume that $A(P, Q)$ is nonempty. The cases when it is empty are shown in Figure 14. In these cases the connectivity number $c = 4g + s + s' + t + u - 6$ equals $-2$ or $-3$ so the assertion of the theorem is automatically true. In all other cases $c \geq -1$, in agreement with the fact that $A(P, Q)$ is nonempty.

By Lemmas 5.2 and 5.3 $A(P, Q)$ is contractible if some boundary circle of $S$ meets only $P$ or only $Q$, or if some boundary circle contains three successive arcs of $P$ or of $Q$, so we may assume that neither of these possibilities occurs.

Lemma 5.5 shows that removing a disk from $S$ increases the connectivity of $A(P, Q)$ by one and Lemma 5.4 that dividing an arc of $P$ or $Q$ into two disjoint arcs separated by a pure arc also increases the connectivity of $A(P, Q)$ by one. Each of these operations also increases $c$ by one, so it therefore suffices to prove the theorem assuming every boundary component meets both $P$ and $Q$ (in particular $s = s'$) and there are no pure arcs ($u = 0$).

The fact that $s = s'$ will be true for all situations which arise in the proof, so the connectivity bound we claim can also be written as

$$c = 2e + t + u - 2$$

where $e = -\chi(S) = 2g - 2 + s$ is the negative of the Euler characteristic. Although we may initially assume $u = 0$, in the course of the proof we will have to consider cases with $u \neq 0$.

We have $A(P, Q) \subset A(P \cup Q)$. The proof of the theorem is by induction on the dimension of $A(P \cup Q)$. The induction will start with the case that this dimension is 0, and the only times this happens with $A(P, Q)$ nonempty are when $S$ is a disk and $P \cup Q$ consists of four components. In these cases one can check directly that the theorem is true.
Let \( f: S^k \to A(P, Q) \) represent an element of \( \pi_k A(P, Q) \), for \( k \leq 2e + t - 2 \). We claim that \( f \) extends to a map \( D^{k+1} \to A(P \cup Q) \), which we shall also call \( f \). Lemma 5.6 tells us that \( A(P \cup Q) \) is contractible in most cases, so in those cases the map certainly extends. The only exceptional case we need to consider is \( S = D^2 \). In this case \( A(P \cup Q) \) is \((2t-5)\)-connected, since \( 2t \) is equal to the number of components of \( P \cup Q \). But here \( c = -2 + t + 0 - 2 = t - 4 \leq 2t - 5 \), so \( f \) extends in this case too.

We may assume \( f \) is simplicial with respect to some PL triangulation of \( D^{k+1} \), and we now want to use a link argument to replace \( f \) by a new map \( D^{k+1} \to A(P, Q) \) which still extends the original \( f \) on \( S^k \). To understand the link of a simplex we will cut the surface open along the arcs defining the simplex. If we cut \( S \) open along a collection of arcs we obtain a number of surfaces \( S_i \), and \( P \) and \( Q \) are cut into collections \( P_i \) and \( Q_i \) of disjoint arcs in \( \partial S_i \). It can happen that one of \( P_i \) and \( Q_i \) is empty. In this case we call \( S_i \) a pure piece of \( S \), and otherwise we call \( S_i \) a mixed piece. Note that the dimension of each \( A(P_i \cup Q_i) \) is strictly less than the dimension of \( A(P \cup Q) \) so we can assume inductively that the theorem is true for each mixed piece: \( A(P_i \cup Q_i) \) is \((2e_i + t_i + u_i - 2)\)-connected.

Since the ‘good’ subspace \( A(P, Q) \subset A(P \cup Q) \) contains no arcs joining \( P \) to itself or \( Q \) to itself, we define such arcs to be bad. A first guess on how to proceed with a link argument would be to define a simplex of \( A(P \cup Q) \) to be bad if all of its vertices are bad arcs. However this does not work. If \( \mu \) is a maximal simplex of \( D^{k+1} \) such that \( \sigma = f(\mu) \) consists entirely of bad arcs, then the link of \( \mu \) must map to the join of the complexes \( A(P_i \cup Q_i) \) for the mixed pieces \( S_i \) of \( S - \sigma \). If this join were \((k - \dim(\mu))\)-connected then the link argument (Corollary 2.2) could be used to finish the proof. Unfortunately the join need not be \((k - \dim(\mu))\)-connected or even nonempty. As an example, consider the situation in Figure 15. Here \( S \) is a surface with one boundary circle, \( P \) and \( Q \) each consist of one arc, \( k = 0 \), \( \mu \) is a 1-simplex, and \( f(\mu) \) consists of two bad arcs \( a_0 \) and \( a_1 \) joining \( P \) to itself. The arc \( a_0 \) cuts off a disk from \( S \) containing \( Q \), and \( a_1 \) lies in the complement of this disk, a subsurface of \( S \) disjoint from \( Q \). The only mixed piece \( S_i \) that occurs when we split \( S \) along \( f(\mu) \) is the disk cut off by \( a_0 \), and this has \( A(P_i \cup Q_i) \) empty.

To avoid this sort of problem we use a more restrictive definition of a bad simplex: \( \sigma \) is bad if all of its arcs are bad and none of its arcs have pure pieces on both sides. Note that a
single bad arc is a bad vertex, since cutting $S$ along a single bad arc produces at most two pieces $S_i$, and if both of these were pure then one of $P$ and $Q$ would be empty.

We cannot apply Corollary 2.2 directly since condition (2) in the definition of badness in Section 2.1 fails here. For example, an edge can have bad endpoints but not be bad itself. However we will use the same strategy of retriangulating the star of $\mu$ and redefining $f$ on this star. We will not be able to eliminate bad simplices right away but instead will aim for the more modest goal of simplifying the complementary pure pieces $S_i$. To measure how complicated a pure piece is we use the number $d(S_i)$ of arcs in a maximal arc system in $A(P_i \cup Q_i)$. (Maximal systems all have the same number of arcs by an Euler characteristic argument.)

If $\mu$ is a simplex of $D^{k+1}$ with bad image $\sigma = f(\mu)$ we define the complexity of $\mu$ to be the ordered pair

$$(d(\sigma), \dim(\mu))$$

where $d(\sigma)$ is the sum of the $d(S_i)$ for the pure pieces $S_i$ that arise when $S$ is cut along $\sigma$. Now let $\mu$ be a bad simplex of $D^{k+1}$ which has maximum complexity $(d, l)$. The restriction of $f$ to the link $lk(\mu) = S^{k-l}$ then maps to the join $J$ of the complexes $A(P_i, Q_i)$ for the mixed pieces and the complexes $A(P_i \cup Q_i)$ for the pure pieces.

Suppose for the moment that $J$ is $(k-l)$-connected, so we can extend this map $f : S^{k-l} \to J$ to a map $F : D^{k-l+1} \to J$. We modify $f$ on the star of $\mu$ to a new map $f_{|\partial \mu} * F$, as in the proof of Lemma 2.1. We claim that simplices with bad image in the new triangulation of the star of $\mu$ have lower complexity than $\mu$, i.e., if $\nu$ is a simplex in the new triangulation of $st(\mu)$ with bad image $\tau$, then either $d(\tau) < d(\sigma)$ or $d(\tau) = d(\sigma)$ and $\dim(\nu) < \dim(\mu)$. To see this, note that $\nu$ is the join of a simplex $\alpha$ in $\partial \mu$ with a simplex $\beta$ mapping to $J$. If $\beta$ is nonempty then since the vertices of $f(\beta)$ are bad they do not lie in any $A(P_i, Q_i)$, so they are contained in the pure pieces associated to the original $\sigma$. These arcs cut pure pieces into pure subpieces, and hence the join $\sigma * f(\beta)$ fails to be a bad simplex. Since $\tau$ is bad, it must be obtained from $\sigma * f(\beta)$ by deleting some of the arcs of $\sigma$, which forces $d(\tau) < d(\sigma)$. In case $\beta$ is empty, so $\nu$ is a simplex of $\partial \mu$, either $d(\tau) < d(\sigma)$ or $\tau = \sigma$ and $\nu$ has smaller dimension than $\mu$ by the maximality of $\mu$.

After a finite number of modifications it will no longer be possible to decrease complexity, at which point we can conclude that there are no bad simplices. In particular there are no bad vertices, so there are no bad arcs in the image of the final $f$ and we are done.

It remains to verify that $J$ is $(k-l)$-connected. Since the dimension of $\sigma$ is at most $l$, it suffices to show $J$ is $k - (q-1)$-connected, where $q$ is the number of arcs in $\sigma$. Since $k \leq 2e + t - 2$, in fact it suffices to show $J$ is $(2e + t - q - 1)$-connected.

If any pure piece $S_i$ is not a disk then $A(P_i \cup Q_i)$ is contractible by Lemma 5.6, which implies that $J$ is contractible. Thus we may assume all pure pieces $S_i$ are disks, and then $A(P_i \cup Q_i)$ is $(u_i - 5)$-connected by Lemma 5.6. By induction we may assume that for each mixed $S_i$ the complex $A(P_i, Q_i)$ is $(2e_i + t_i + u_i - 2)$-connected. Since connectivity plus two is additive for joins the connectivity of $J$ is at least

$$\sum_{\text{mixed } S_i} (2e_i + t_i + u_i) + \sum_{\text{pure } S_i} (u_i - 3) - 2$$
If we let \( p \) be the number of pure pieces, this connectivity bound can be rewritten as
\[
2 \sum_{\text{mixed } S_i} e_i + \sum_i t_i + \sum_i u_i - 3p - 2 \quad (*)
\]
Since the pure pieces are all disks, with \( e_i = -\chi(S_i) = -1 \), we have
\[
e = (\sum_i e_i) + q = \sum_{\text{mixed } S_i} e_i - p + q
\]
Since cutting \( S \) along bad arcs only introduces pure boundary arcs we have \( \sum_i t_i = t \). Cutting along a bad arc creates two pure boundary arcs, one on either side of the cut, so \( \sum_i u_i = 2q \) since initially we had no pure arcs \((u = 0)\). In light of these observations the expression \((*)\) above giving a lower bound for the connectivity of \( J \) becomes
\[
2(e + p - q) + t + 2q - 3p - 2 = 2e + t - p - 2
\]
Since no arc of \( \sigma \) lies in more than one pure \( S_i \) and each pure \( S_i \) is a disk with at least three arcs of \( \sigma \) in its boundary, we have \( q \geq 3p \). This implies \( q > p \) if \( p > 0 \), and we also have \( q > p \) when \( p = 0 \) since \( q \geq 1 \). We conclude that
\[
\text{connectivity}(J) \geq 2e + t - q - 1
\]
as desired. \( \square \)

Let \( A^0(P, Q) \) be the subcomplex of \( A(P, Q) \) formed by coconnected arc systems. The connectivity result we will need in the next section is the following:

**Proposition 5.7.** If the arc components of \( P \) and \( Q \) alternate in \( \partial S \) then \( A^0(P, Q) \) is \((2g + s' - 3)\)-connected, where \( s' \) is the number of components of \( \partial S \) containing points of \( P \) or \( Q \).

Observe that the quantity \( 2g + s' - 3 \) is \( e - 1 \) where \( e \) is minus the Euler characteristic of the surface obtained from \( S \) by filling in the boundary circles disjoint from \( P \) and \( Q \) with disks.

**Proof.** The proof is a link argument that deduces the high connectivity of \( A^0(P, Q) \) from that of \( A(P, Q) \). Observe first that \( 2g + s' - 3 \leq 4g + s + s' + t - 6 \) in all cases except when \( g = 0 \) and \( s = t = 1 \), but the connectivity statement for \( A^0(P, Q) \) is vacuous in this case. For the link argument the bad simplices of \( A(P, Q) \) are those for which the dual graph of the corresponding arc system has no edges that are loops. Let \( \sigma \) be a bad \( k \)-simplex, corresponding to a system of \( k + 1 \) arcs cutting \( S \) into surfaces \( S_1, \ldots, S_j \), where \( j \geq 2 \) by the badness condition. By induction on the lexicographically ordered triple \((g, s, t)\) we may assume that the complex \( A^0(P_i, Q_i) \) for the surface \( S_i \) is \((e_i - 1)\)-connected. The induction starts with the cases \( g = 0, s \leq 2 \), when the result is obviously true since the assertion is vacuous when \( s' = 1 \) and \( A^0(P, Q) \) is nonempty when \( s' = 2 \). The complex \( G_\sigma \) is the join of the complexes \( A^0(P_i, Q_i) \) so its connectivity is at least \( \sum_i(e_i - 1) + 2j - 2 = \sum_i e_i + j - 2 \). We have \( e = \sum_i e_i + k + 1 \) so the connectivity of \( G_\sigma \) is at least \( e - k - 1 + j - 2 \), which is at least \( e - k - 1 \) since \( j \geq 2 \). Thus the hypotheses of Corollary 2.2 are satisfied and the link argument is finished. \( \square \)
6. Chains and tethered chains

To prove that \( Ch(S) \) is highly-connected we will not use the forgetful map \( Ch(S) \to C^0(S) \) directly, but instead first embed \( Ch(S) \) into a larger complex \( MCh(S) \) of multi-chains. The forgetful map extends to \( MCh(S) \) and we prove that the fibers of this extension are sufficiently highly-connected to ensure that \( MCh(S) \) is \((g-2)\)-connected. We then finish with a link argument to show that \( Ch(S) \) is \((g-3)/2\)-connected.

The complex \( MCh(S) \) has the same vertices as \( Ch(S) \) but many more higher-dimensional simplices. For example an edge is added between two chains with the same \( a \)-curve if the \( b \)-curves either become parallel or form a coconnected arc system when an annular neighborhood of \( a \) is removed from \( S \). The \( b \)-curves may intersect inside this neighborhood. (See Figure 16.)

In general a set of chains \( c_{ij} = (a_i, b_{ij}) \) forms a simplex in \( MCh(S) \) if the \( a_i \)'s form a coconnected curve system and have disjoint annular neighborhoods \( N(a_i) \) satisfying the following two conditions:

- Each \( b_{ij} \) is disjoint from \( N(a_k) \) for \( k \neq i \), and \( b_{ij} \) intersects \( N(a_i) \) in an arc going from one boundary circle of \( N(a_i) \) to the other. There is no restriction on how the \( b_{ij} \)'s intersect inside \( N(a_i) \).
- Outside \( \cup_i N(a_i) \) any two \( b_{ij} \)'s are either disjoint or coincide, and their union is a coconnected arc system in \( S - \cup_i N(a_i) \).

The complex \( MCh(S) \) contains \( Ch(S) \) as the sets of chains \( (a_i, b_{ij}) \) with only one \( b_{ij} \) for each \( a_i \). Unlike \( Ch(S) \), \( MCh(S) \) is infinite dimensional since there is no bound on the number of \( b_{ij} \)'s that can be paired with one \( a_i \). For example one can start with a single \( b_i \) and apply repeated Dehn twists along \( a_i \) to it to obtain an arbitrarily large set of \( b_{ij} \)'s paired with \( a_i \).

**Proposition 6.1.** The complex \( MCh(S) \) is \((g-2)\)-connected.

**Proof.** The forgetful map \( f: MCh(S) \to C^0(S) \) sends a chain \((a_i, b_{ij})\) to the curve \( a_i \). We consider the induced map \( \tilde{f}: \hat{MCh}(S) \to \hat{C^0}(S) \) on the posets of simplices. For a simplex \( \alpha = \{a_0, \ldots, a_k\} \) of \( C^0(S) \) let \( F_\alpha = \tilde{f}^{-1}(\alpha) \). By Proposition 2.8 it will suffice to show that \( F_\alpha \) is \((g - k - 2)\)-connected.

In order to do this we consider the subsurface \( S_\alpha \) of \( S \) obtained by deleting the interiors of the annuli \( N(a_i) \). The boundary curves of \( N(a_i) \) form pairs of boundary curves \( a_i' \) and \( a_i'' \) of \( S_\alpha \). To each collection of chains \((a_i, b_{ij})\) forming a simplex \( \gamma \) of \( MCh(S) \) we can associate the
system of arcs $\gamma_\alpha = \cup_{i,j} b_{ij} \cap S_\alpha$ in $S_\alpha$. These arc systems $\gamma_\alpha$ lie in the poset $A_\alpha$ consisting of coconnected systems of arcs in $S_\alpha$ joining an $a_i'$ to the corresponding $a_i''$, where there is at least one such arc for each $i$.

The poset map $g: F_\alpha \to A_\alpha$ sending $\gamma$ to $\gamma_\alpha$, is a homotopy equivalence because its fibers $g_{\geq \sigma}$ are contractible. (If $\gamma_0$ is any element of $g^{-1}(\sigma)$, the poset maps $\gamma \mapsto \gamma \cup \gamma_0 \mapsto \gamma_0$ give a contraction.)

Thus we are reduced to showing that $A_\alpha$, which is a complex of arc systems on a surface $S_\alpha$ of genus $g - k - 1$, is $(g - k - 2)$-connected. To simplify notation we can forget about the original surface $S$ and relabel $S_\alpha$ as $S$ and $A_\alpha$ as $A_k$, where the subscript $k$ indicates that there are $k + 1$ pairs of boundary circles of $S$ that are being joined by arcs. If the genus of the new $S$ is $g$, we wish to show $A_k$ is $(g - 1)$-connected.

We reduce this to the case $k = 0$ using induction and the poset map $f_0: A_k \to A_0$ which ignores all arcs except those from $a_i'$ to $a_i''$. Since we are dealing with coconnected arc systems, each fiber $f_0^{-1}(\sigma)$ can be identified with $A_{k-1}$ for the surface obtained by cutting $S$ along the arc system $\sigma$. Cutting $S$ along the $m + 1$ arcs in an $m$-simplex can decrease the genus by at most $m$, as cutting along the first arc merely connects two boundary circles together without reducing the genus. Therefore by induction the fiber over an $m$-simplex is at least $(g - m - 1)$-connected. The result then follows from Proposition 2.8 provided that $A_0$ is $(g - 1)$-connected.

The complex $A_0$ is the poset of simplices in the complex $A^0(P,Q)$ of coconnected systems of arcs joining one component $P$ of $S$ to another component $Q$. By Theorem 5.7 $A(P,Q)$ is $(2g - 1)$-connected, and $2g - 1 \geq g - 1$. □

We now show $Ch(S)$ is highly-connected using a link argument.

**Theorem 6.2.** The complex $Ch(S)$ of chains on $S$ is $(g - 3)/2$-connected.

**Proof.** Regarding $Ch(S)$ as a subcomplex of $MCh(S)$, the bad simplices $\sigma$ are formed by sets of chains $(a_i, b_{ij})$ where there are at least two $b_{ij}$’s for each $a_i$. In particular, a bad simplex must have dimension at least one. The complex $G_\sigma$ is the complex $Ch(S_\sigma)$ where $S_\sigma$ is obtained from $S$ by first removing neighborhoods $N(a_i)$ of the $a_i$’s and then cutting open along the remaining arcs of the $b_{ij}$’s. If $\sigma$ is a 1-simplex then $S_\sigma$ has genus at most two less than $S$, and this “two” cannot be improved to “one”. In general, if $\sigma$ is a $k$-simplex then the genus of $S_\sigma$ is at least $g - k - 1$.

We wish to find an increasing function $n = \varphi(g)$ so that $Ch(S)$ is $n$-connected. To apply Corollary 2.2 we must first have $\varphi(g) \leq g - 2$, the connectivity of $MCh(S)$. There is also the inductive condition on the connectivity of $Ch(S_\sigma)$, which is the inequality $\varphi(g) - k \leq \varphi(g - k - 1)$ since $S_\sigma$ has genus at least $g - k - 1$. When $k = 1$ this says $\varphi(g) \leq \varphi(g - 2) + 1$. A linear function $\varphi(g) = Ag + B$ satisfies this if $A \leq 1/2$ so we take $A = 1/2$. The more general condition $\varphi(g) - k \leq \varphi(g - k - 1)$ then becomes $\frac{g}{2} + D - k \leq \frac{g - k - 1}{2} + D$ which reduces to $k \geq 1$ so this holds. To start the induction, the constant $D = -3/2$ works for $g = 0$, 1 since $Ch(S)$ is empty when $g = 0$ and nonempty when $g = 1$. Finally, the inequality $\varphi(g) \leq g - 2$ for $\varphi(g) = \frac{g - 3}{2}$ reduces to $g \geq 1$, which we may assume since the theorem is vacuous when $g = 0$. □
Example 6.3. Consider the case that $S$ is closed of genus 2, so $Ch(S)$ is one-dimensional. A chain $(a, b)$ in $S$ has neighborhood bounded by a separating curve $c = c(a, b)$. The link of $(a, b)$ in $Ch(S)$ consists of all chains $(a', b')$ in the genus one surface on the other side of $c$. These chains also have $c(a', b') = c$, so it follows that all chains $(a, b)$ in this connected component of $Ch(S)$ have the same curve $c(a, b)$. The connected components of $Ch(S)$ thus correspond to nontrivial separating curves on $S$. Each connected component is the join of two copies of the infinite zero-dimensional complex $Ch(S_{1,1})$. Thus $Ch(S)$ is not homotopy equivalent to a wedge of spheres of a single dimension, in contrast with the situation for the curve complexes $C(S)$ and $C^0(S)$. Note also that the connectivity bound $(g - 3)/2$ is best possible in this case, where it asserts only that $Ch(S)$ is nonempty.

Remark 6.4. As a partial generalization of this example one can say that for $S$ a surface of arbitrary genus $g \geq 1$ the group $H_{g-1}(Ch(S))$ is free abelian of infinite rank. This follows as in Remark 4.9 by embedding $g$ disjoint copies of $S_{1,1}$ in $S$, which gives an embedding of the join of $g$ copies of the infinite discrete set $Ch(S_{1,1})$ in $Ch(S)$. This join has dimension $g - 1$, the same dimension as $Ch(S)$, so the embedding of the join is injective on $H_{g-1}$.

Now we move on to the complex $TCh(S)$ of tethered chains. As in the case of $Ch(S)$ we will not use the forgetful map $TCh(S) \to Ch(S)$ directly, but rather embed $TCh(S)$ in a larger complex $MTCh(S)$ of multi-tethered chains and use the extended forgetful map $MTCh(S) \to Ch(S)$ to show that $MTCh(S)$ is highly-connected. Then a link argument will show that $TCh(S)$ is highly-connected.

To define the complex $MTCh(S)$ first choose points $p_1, \cdots, p_m$ in $\partial S$. A vertex of $MTCh(S)$ is then an isotopy class of tethered chains $(c, t)$ in $S$ where the tether $t$ attaches to any one of the points $p_i$. A set of tethered chains $(c_i, t_i)$ spans a simplex if the $c_i$'s are either disjoint or coincide, and the tethers $t_i$ are all disjoint from each other and from the $c_i$’s except at their endpoints. There is no condition of coconnectedness as there was for multi-chains.

Proposition 6.5. $MTCh(S)$ is $(g - 3)/2$-connected.

Proof. We map $MTCh(S)$ to $Ch(S)$ by ignoring the tethers. The fiber over the barycenter of a simplex $\sigma$ of $Ch(S)$ consists of all systems of multi-tethers for $\sigma$. Choose one such system consisting of a single tether $t_i$ to each chain $c_i$ in $\sigma$. We can deform this barycentric fiber into the star of this tethered system by a surgery flow. The surgeries are performed first using the tether $t_1$ until all tethers are disjoint from $t_1$, then using $t_2$, and so on.

Thus the barycentric fiber over $\sigma$ is contractible, so the projection $MTCh(S) \to Ch(S)$ is a homotopy equivalence by Proposition 2.8. \qed

Now we look at the subcomplex $TCh(S)$ of $MTCh(S)$ where there is only one tether to each chain, and where the other ends of the tethers are at the points $p_i$. For the application to homology stability we only need the case of a single $p_i$, but for the proof of high connectivity the more general case is needed.

Theorem 6.6. $TCh(S)$ is $(g - 3)/2$-connected.
Spectral sequences and stability

7.1. Non-closed surfaces with a fixed number of boundary components. Let $S$ be a surface of genus $g$ with $s \geq 1$ boundary components, and let $M_{g,s}$ be the mapping class group of $S$, where diffeomorphisms and isotopies between them are required to restrict to the identity on each boundary circle.

**Theorem 7.1.** For each $s \geq 1$ the stabilization $H_i(M_{g-1,s}) \to H_i(M_{g,s})$ is an isomorphism for $g > 2i + 2$ and a surjection for $g = 2i + 2$.

**Proof.** Consider the action of $M_{g,s}$ on the tethered chain complex $TCh(S)$. This action is transitive on simplices of each dimension, and the stabilizer of a system of $k$ tethered chains is isomorphic to $M_{g-k,s}$. The quotient $TCh(S)/M_{g,s}$ is the $\Delta$-complex with a single $i$-simplex in each dimension $i \leq g - 1$, so it is $(g - 2)$-connected, as we saw in the proof of Theorem 3.3, item (4). An edge of $TCh(S)$ corresponds to a system of two tethered chains. A small neighborhood of this system forms a surface of genus 2 with one boundary component, and there is a diffeomorphism of $S$ supported on this neighborhood which interchanges the two chains and sends the first tether to the second. Another diffeomorphism interchanges the
chains and sends the second tether to the first. Thus the hypotheses for deducing homology
stabilization are satisfied and the theorem follows by the argument in Example 1.2. □

7.2. Non-closed surfaces with a varying number of boundary components.

To deal with closed surfaces we will need to know that in the non-closed case, the stable homology
does not depend on the number of boundary components. This will follow by combining the
previous theorem with a simpler variant of it that involves the stabilization $M_{g-1,s+1} \to M_{g,s}$
obtained by gluing a pair of pants to $S_{g-1,s+1}$ along two of its boundary curves, assuming
$s \geq 1$.

**Theorem 7.2.** For each $s \geq 1$ the stabilization $H_i(M_{g-1,s+1}) \to H_i(M_{g,s})$ is an isomorphism
for $g > 2i + 1$ and a surjection for $g = 2i + 1$.

**Proof.** This is very similar to the proof of Theorem 7.1, but with the tethered chain complex
replaced by the complex $TC_0(S)$ of tethered coconnected curve systems and Example 1.2
replaced by Example 1.1. To begin the argument we need to show $TC_0(S)$ is highly connected.
We first we embed $TC_0(S)$ into a multi-tethered version and note that the fibers of the
forgetful map to $C_0(S)$ are contractible by surgery. A straightforward link argument then
allows us to conclude that $TC_0(S)$ is $(g-2)$-connected.

The vertex stabilizers of the action of $M_{g,s}$ on $TC_0(S)$ are copies of $M_{g-1,s+1}$ and more
generally the stabilizers of $k$-simplices are copies of $M_{g-k-1,s+k+1}$. Given a system of two
tethered curves in $TC_0(S)$, there is a diffeomorphism of $S$ supported on a neighborhood of
the system taking the first tethered curve to the second, or the second to the first. All of the
conditions for showing homology stability are thus satisfied and the conclusion follows as in
Example 1.1. □

**Corollary 7.3.** For $s \geq 1$ the $s$-stabilization $H_i(M_{g,s}) \to H_i(M_{g,s+1})$ is an isomorphism
for $g \geq 2i+1$.

**Proof.** The $g$-stabilization $M_{g-1,s} \to M_{g,s}$ factors as the composition of the two stabilizations
$M_{g-1,s} \to M_{g-1,s+1} \to M_{g,s}$ by first attaching a pair of pants along one boundary curve, then
attaching another pair of pants to the other two boundary curves of the first pair of pants.
The second stabilization is an isomorphism on $H_i$ for $g \geq 2i+2$ and the composition is a
surjection in the same range, so the first stabilization is also a surjection in this range, which
we can write as $g-1 \geq 2i+1$. The $s$-stabilization is always injective since it has a one-sided
inverse obtained by filling in one of the two new boundary curves with a disk. □

7.3. Closed surfaces. It remains to deal with the projection $M_{g,1} \to M_{g,0}$ induced by filling
in the boundary circle of $S_{g,1}$ with a disk.

**Theorem 7.4.** The projection $H_i(M_{g,1}) \to H_i(M_{g,0})$ is an isomorphism for $g > 2i + 2$
and a surjection for $g = 2i + 2$.

**Proof.** The complex of tethered chains is not defined when $s = 0$ but there is a variant of
the complex of chains $Ch(S)$ that we can use instead. The action of $M_{g,s}$ on $Ch(S)$ is again
transitive on simplices of each dimension. The stabilizer of a system of $k$ chains contains
$M_{g-k,s+k}$ but is actually slightly bigger than this for two reasons. First, elements in the
stabilizer can permute the chains in the system, and second, they can reverse the orientations of the two curves in a chain. The latter problem can be taken care of by considering the oriented version $Ch^\pm(S)$ of $Ch(S)$ in which each $a$-curve in a chain has a chosen orientation. This is also $(g - 3)/2$-connected. The proof of this is the same as for $Ch(S)$ once one knows that the oriented version of the complex $C^0(S)$ of coconnected curve systems has the same connectivity as $C^0(S)$ itself, which is immediate from Lemma 2.3, namely, choose an arbitrary orientation for each isotopy class of coconnected curves and then the two possible orientations correspond to the labels $+$ and $-$. 

To deal with permutations of chains we can use the following general construction. Given a simplicial complex $Z$, let $\Delta(Z)$ be the $\Delta$-complex whose $k$-simplices are all the simplicial maps $\Delta^k \to Z$. These are not required to be injective. Thus $\Delta(Z)$ has the same vertices as $Z$, but has more simplices in each higher dimension. For example, it has two 1-simplices for each 1-simplex of $Z$, corresponding to the two orderings of the vertices of this simplex, and it also has a 1-simplex for each vertex of $Z$, corresponding the constant map $\Delta^1 \to Z$ with image this vertex. There is a natural projection $\Delta(Z) \to Z$ and it is a classical fact that this induces an isomorphism on homology; see for example Theorem 4.3.8 in [21]. Here is a sketch of a proof: Since homology commutes with direct limits, it suffices to assume that $Z$ is a finite complex, and this case can be handled by induction on the number of (open) simplices. For the induction step one uses Mayer-Vietoris sequences and the five lemma, decomposing $Z$ as the union of a top-dimensional simplex and the complement of the interior of this simplex. This reduces the proof to the case that $Z$ itself is a simplex. In this special case $\Delta(Z)$ is contractible via the homotopy that cones off each simplex to a chosen vertex of $Z$.

We will apply the spectral sequence argument to the action of $M_{g,0}$ on $X = \Delta(Ch^\pm(S))$. Since $X$ is not a simplicial complex but only a $\Delta$-complex, we first observe that the spectral sequence works equally well in this extra generality.

We now check that $X/M_{g,0}$ is $(g - 2)$-connected. Since $M_{g,0}$ acts transitively on simplices of $Ch^\pm(S)$ we can identify $X/M_{g,0}$ with $\Delta(\Delta^{g-1})/\Sigma_g$ where the symmetric group $\Sigma_g$ acts on $\Delta^{g-1}$ by permuting its vertices. The $(g - 2)$-skeleton of $\Delta(\Delta^{g-1})/\Sigma_g$ is contained in $\Delta(\Delta^{g-2})/\Sigma_{g-1} \subset \Delta(\Delta^{g-1})/\Sigma_g$ so it will suffice to show that $\Delta(\Delta^{g-2})/\Sigma_{g-1}$ is contractible in $\Delta(\Delta^{g-1})/\Sigma_g$. This holds since the inclusion map $\Delta(\Delta^{g-2}) \to \Delta(\Delta^{g-1})$ extends to a map of the cone $C(\Delta(\Delta^{g-2})) \to \Delta(\Delta^{g-1})$ by coning off to the last vertex of $\Delta^{g-1}$, and this map passes to a quotient map $C(\Delta(\Delta^{g-2})/\Sigma_{g-2}) \to \Delta(\Delta^{g-1})/\Sigma_{g-1}$.

Condition 3 at the beginning of Section 1 holds for the degenerate 1-simplices in $X$ but fails for the nondegenerate 1-simplices since there is no diffeomorphism of $S_{g,0}$ moving one chain onto another disjoint chain and supported in a neighborhood of the two chains. However there is a weakening of Condition 3 that is satisfied and is strong enough to make the argument for injectivity of the differential $d: H_i(M_{g-1,1}) \to H_i(M_{g,0})$ still work. If we enlarge the neighborhood of the two chains by adding a neighborhood of an arc joining them, then there is a diffeomorphism interchanging the two chains supported in the enlarged neighborhood. By $s$-stability we know that classes in $H_i$ of the stabilizer $M_{g-2,2}$ of the two chains come from classes in $H_i$ of the subgroup $M_{g-2,1}$ stabilizing the connecting arc as well as the two chains, provided that $g - 2 \geq 2i + 1$. This suffices to deduce injectivity of the differential.
$d: H_i(M_{g-1,1}) \to H_i(M_{g,0})$. For surjectivity we do not need Condition 3, and the inequality $g \geq 2i + 2$ suffices.

The differential $d$ factors as the composition of the two stabilizations $H_i(M_{g-1,1}) \to H_i(M_{g,1}) \to H_i(M_{g,0})$, where the second stabilization is the one in the statement of the present theorem. The latter stabilization is surjective when the composition is surjective, i.e., when $g \geq 2i + 2$, and it is an isomorphism when the first stabilization and the composition are isomorphisms, i.e., when $g > 2i + 2$. □

References

[18] Andrew Putman and Steven Sam, Representation stability and finite linear groups, arXiv:1408.3694