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Incompressible surfaces in 2-bridge knot complements

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To each rational number p/q, with q odd, there is associated the 2-bridge knot $K_{p/q}$ shown in Fig. 1.

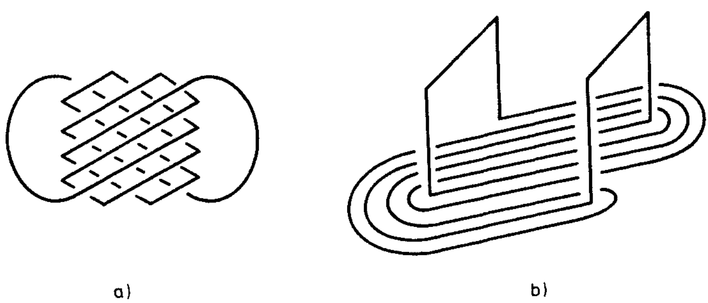


Fig. 1. The 2-bridge knot $K_{p/q}$

In (a), the central grid consists of lines of slope $\pm p/q$, which one can imagine as being drawn on a square "pillowcase". In (b) this "pillowcase" is punctured and flattened out onto a plane, making the two "bridges" more evident. The knot drawn is $K_{3/5}$, which happens to be the figure eight knot. (We assume q odd in order to get a knot rather than a two-component link.) The double cover of S^3 branched along $K_{p/q}$ is the lens space $L_{q,p}$. With this observation, attributed in [16] to Seifert, the isotopy classification of 2-bridge knots follows easily from the classification [14] of oriented lens spaces: $K_{p/q} = K_{p'/q'}$ if and only if $q'=q$ and $p' \equiv p^{\pm 1} \pmod q$. Basic references for 2-bridge knots are [2, 16, 17].

We shall derive in this paper the isotopy classification of the incompressible surfaces, orientable or not, in $S^3 - K_{p/q}$. As an application, we obtain some information about the manifold resulting from Dehn surgery on $K_{p/q}$: Excluding the cases when $K_{p/q}$ is a torus knot (Dehn surgery on torus knots was completely analyzed in [10]), every Dehn surgery on $K_{p/q}$ yields an irreducible manifold, and all but finitely many Dehn surgeries yield non-Haken (i.e., not sufficiently large) non-Seifert-fibered manifolds. The case of the figure eight knot, $K_{3/5}$, was previously worked out in [18] and gave the first known examples of non-Haken, non-Seifert-fibered irreducible 3-manifolds. Another generalization of the case of $K_{3/5}$, to punctured-torus bundles, can be found in [3] and [5].

A version of this paper was circulated in preprint form in 1979. Since that time a number of improvements in techniques have been found [5, 7, 12], and we have taken advantage of these subsequent developments in revising this paper for publication. Extensions of some of our results to wider classes of knots and links can be found in [6, 9, 11].

§ 1. Results

There are two common definitions of incompressibility for a surface $S \neq S^2$, D^2, $\mathbb{R}P^2$ embedded in a 3-manifold M, with $S \cap \partial M = \partial S$. We shall use the weaker one: S is *incompressible* in M if for each disc $D \subset M$ with $D \cap S = \partial D$ there is a disc $D' \subset S$ with $\partial D' = \partial D$. The (obviously) stronger condition is that $\pi_1 S \to \pi_1 M$ be injective. If the normal bundle of S in M is trivial, then π_1-injectivity is equivalent to incompressibility, by the loop theorem. If the normal bundle of S in M is non-trivial, then $\pi_1 S \to \pi_1 M$ injective is equivalent to the incompressibility of $\tilde{S}$, the boundary of a tubular neighborhood of S in M. Clearly, $\tilde{S}$ incompressible implies S incompressible. For an example of an incompressible surface $S \subset M$ with $\pi_1 S \to \pi_1 M$ not injective, let M be a lens space $L_{q,p}$ with q even. The non-trivial element of $H_2(L_{q,p}; Z_2)$ is represented by an embedded non-orientable surface. Such a surface S of minimal genus must be incompressible. If S were $\mathbb{R}P^2$, $L_{q,p}$ would be $\mathbb{R}P^3$. Hence $\pi_1 S$ is infinite if $q > 2$, and $\pi_1 S \to \pi_1 L_{q,p}$ is not injective. (It follows from the results of this paper that for 2-bridge knot complements, incompressibility is equivalent to π_1-injectivity.)

S is *∂-incompressible* if for each disc $D \subset M$ with $D \cap S = \partial_+ D$ and $D \cap \partial M = \partial_- D$ (where $\partial_+ D \cup \partial_- D = \partial D$ and $\partial_+ D \cap \partial_- D = S^0$), there is a disc $D' \subset S$ with $\partial_+ D' = \partial_+ D$ and $\partial_- D' \subset \partial S$. We recall an elementary fact (see [W], Lemma 1.10 for a proof): In an irreducible orientable 3-manifold whose boundary consists of tori (such as a knot complement), an orientable incompressible surface is either ∂-incompressible or a ∂-parallel annulus. ($S \subset M$ is *∂-parallel* if S can be isotoped into ∂M rel ∂S.)

Our classification of the incompressible, ∂-incompressible surfaces in $S^3 - K_{p/q}$ will be in terms of the continued fraction expansions of p/q,

$$p/q = r + [b_1, \ldots, b_k] = r + \cfrac{1}{b_1 - \cfrac{1}{b_2 - \ddots \, - \cfrac{1}{b_k}}} \qquad r, b_i \in \mathbb{Z}$$

As is well-known (see [2, 17]), if $p/q = r + [b_1, \ldots, b_k]$, then $K_{p/q}$ is the boundary of the surface obtained by plumbing together k bands in a row, the i^{th} band having b_i half-twists (right-handed if $b_i > 0$ and left-handed if $b_i < 0$).

There are two essentially different ways of performing each plumbing of two adjacent bands. One way of describing this choice is to say that instead of using one of the horizontal plumbing squares shown in Fig. 2, we could use the complement of this square in the horizontal plane containing it, compactified by a point at ∞. (Thus we are now regarding S^3 as the 2-point compactification of $S^2 \times \mathbb{R}$, with the spheres $S^2 \times \{*\}$ being horizontal.)

If we include for each plumbing *both* of these complementary horizontal plumbing squares, we obtain a certain branched surface $\Sigma[b_1, \ldots, b_k]$; see Fig. 3. $\Sigma[b_1, \ldots, b_k]$ carries a large number of (not necessarily connected) surfaces, labelled $S_n(n_1, \ldots, n_{k-1})$, where $n \geqq 1$ and $0 \leqq n_i \leqq n$. By definition, $S_n(n_1, \ldots, n_{k-1})$ consists of n parallel sheets running close to the vertical portions of each band of $\Sigma[b_1, \ldots, b_k]$, which bifurcate into n_i parallel copies of the i^{th} inner plumbing square and $n-n_i$ parallel copies of the i^{th} outer plumbing square. For example, when $n=1$, the surfaces $S_1(n_1, \ldots, n_{k-1})$ ($n_i=0$ or 1) are just the 2^{k-1} plumbings of the original k bands.

In order to have all the surfaces $S_n(n_1, \ldots, n_{k-1})$ for different $\Sigma[b_1, \ldots, b_k]$'s lying in a single copy of $S^3 - K_{p/q}$, we choose a fixed position for $K_{p/q}$, say the

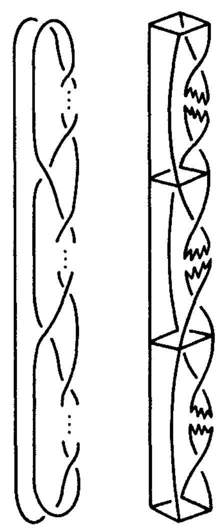


Fig. 2

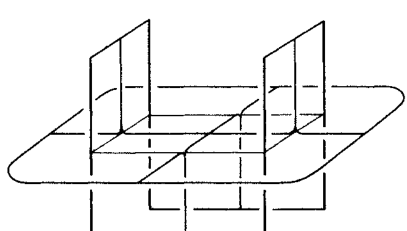


Fig. 3

one shown in Fig. 1b, then reposition $\Sigma[b_1, \ldots, b_k]$ by a level-preserving isotopy of S^3 which deforms $\partial\Sigma[b_1, \ldots, b_k]$ onto this fixed $K_{p/q}$.

Theorem 1. (a) *A closed incompressible surface in $S^3 - K_{p/q}$ is a torus isotopic to the boundary of a tubular neighborhood of $K_{p/q}$.*

(b) *A non-closed incompressible, ∂-incompressible surface in $S^3 - K_{p/q}$ is isotopic to one of the surfaces $S_n(n_1, \ldots, n_{k-1})$ carried by $\Sigma[b_1, \ldots, b_k]$, for some continued fraction expansion $p/q = r + [b_1, \ldots, b_k]$ with $|b_i| \geqq 2$ for each i.*

(c) *The surface $S_n(n_1, \ldots, n_{k-1})$ carried by $\Sigma[b_1, \ldots, b_k]$ is incompressible and ∂-incompressible if and only if $|b_i| \geqq 2$ for each i.*

(d) *Surfaces $S_n(n_1, \ldots, n_{k-1})$ carried by distinct $\Sigma[b_1, \ldots, b_k]$'s with $|b_i| \geqq 2$ for each i are not isotopic.*

(e) *The relation of isotopy among the surfaces $S_n(n_1, \ldots, n_{k-1})$ carried by a given $\Sigma[b_1, \ldots, b_k]$ with $|b_i| \geqq 2$ for each i is generated by:*

(*) *$S_n(n_1, \ldots, n_{i-1}, n_i, \ldots, n_{k-1})$ is isotopic to $S_n(n_1, \ldots, n_{i-1}+1, n_i+1, \ldots, n_{k-1})$ if $b_i = \pm 2$. (When $i=1$ this means $S_n(n_1, n_2, \ldots, n_{k-1})$ is isotopic to $S_n(n_1+1, n_2, \ldots, n_{k-1})$, and similarly when $i=k$.)*

Remarks. 1) For 2-bridge links $K_{p/q}$, q even, Theorem 1 applies to those surfaces meeting both components of the link in the same number of sheets. To treat the other incompressible surfaces, the Diagram in Fig. 4 below must be modified; see [6].

2) R. Riley [15] has shown that representations of a 2-bridge knot group in $PSL_2(\mathbb{C})$ taking meridians to parabolic elements are conjugate to represen-

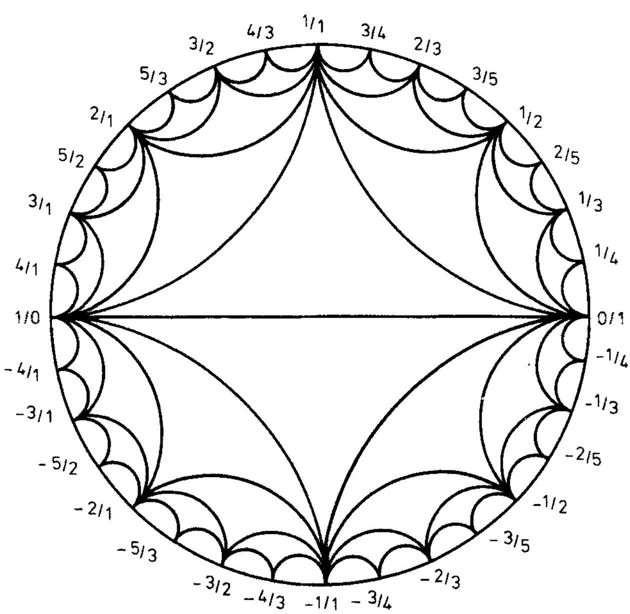


Fig. 4. The diagram

tations in $PSL_2(\mathcal{O})$, where $\mathcal{O} \subset \mathbb{C}$ is the ring of algebraic integers. Theorem 1(a), together with a theorem of Bass [1], implies Riley's result.

The main idea of the proof of Theorem 1 is to associate to an incompressible, ∂-incompressible surface in $S^3 - K_{p/q}$ an edge-path from 1/0 to p/q in the Diagram shown in Fig. 4. By definition, there is an edge in the Diagram joining fractions a/b and c/d whenever $ad-bc=\pm 1$. The edge from a/b to c/d is the long side of triangle whose third vertex is $(a+c)/(b+d)$. This gives a simple inductive rule for labeling the vertices.

Remark. This Diagram comes from the action of $PSL_2\mathbb{Z}$ on the hyperbolic plane: the group of symmetries of a tiling by triangles of this same combinatorial type is $PSL_2\mathbb{Z}$. We have distorted this tiling to space the vertices of the triangles more evenly.

An edge-path from 1/0 to p/q in the Diagram corresponds uniquely to a continued fraction expansion $p/q=r+[b_1,\dots,b_k]$, where the partial sums $p_i/q_i=r+[b_1,\dots,b_i]$ are the successive vertices of the edge-path. At the vertex p_{i-1}/q_{i-1} the path turns left or right across $|b_i|$ triangles, left if $b_i>0$ and right if $b_i<0$.

The criterion $|b_i|\geqq 2$ for incompressibility and ∂-incompressibility is exactly that the associated edge-path be *minimal* in the sense that no edge is immediately retraced and no two edges of one triangle are traversed in succession. Minimal edge-paths from 1/0 to p/q are contained in a finite subcomplex of the Diagram, of the form

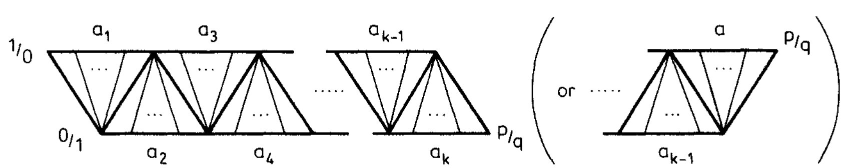


Fig. 5

where the numbers a_i, indicating the number of smaller triangles in each larger triangle, are determined by the unique continued fraction expansion

$$p/q=[a_1,-a_2,a_3,-a_4,\dots,\pm a_k]=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{}{\ddots+\cfrac{1}{a_k}}}},\qquad \begin{array}{l} a_i>0 \\ a_k>1. \end{array}$$

(We have reduced $p/q \bmod 1$, so that $0<p/q<1$.) Clearly, a minimal edge-path in Fig. 5 can involve only the heavy lines which form the large triangles. Then it is not hard to see that the number of minimal edge-paths from 1/0 to p/q is the number φ_k defined recursively by:

$$\varphi_i = \begin{cases} \varphi_{i-2} + \varphi_{i-1}, & a_i > 1 \\ \varphi_{i-3} + \varphi_{i-2}, & a_i = 1, \end{cases}$$

$$\varphi_0 = \varphi_{-1} = 1, \quad \varphi_{-2} = 0.$$

For example, if each $a_i > 1$ then $\{\varphi_i\}$ is the Fibonacci series.

In particular, the number of minimal edge-paths from 1/0 to p/q is finite, so there are, up to isotopy, only finitely many incompressible, ∂-incompressible surfaces in $S^3 - K_{p/q}$ with a given number n of sheets. (Namely, $\Sigma[b_1, \ldots, b_k]$ carries $(n+1)^{l-1}$ isotopy classes of incompressible, ∂-incompressible n-sheeted surfaces, where l is the number of b_i's with $|b_i| > 2$.)

The question of which n-sheeted surfaces $S_n(n_1, \ldots, n_{k-1})$ are connected is slightly subtle. We pursue this only far enough to obtain:

Proposition 1. *Consider the surfaces $S_n(n_1, \ldots, n_{k-1})$ carried by a given $\Sigma[b_1, \ldots, b_k]$. Then:*

(1) *If all b_i's are even, $S_n(n_1, \ldots, n_{k-1})$ is connected only when $n = 1$.*

(2) *If at least one b_i is odd, each two-sheeted surface $S_2(n_1, \ldots, n_{k-1})$ is connected.*

(3) *There exist connected n-sheeted surfaces $S_n(n_1, \ldots, n_{k-1})$ with $n > 2$ if and only if at least two b_i's are odd, in which case there exist connected $S_n(n_1, \ldots, n_{k-1})$'s for all n.*

The single-sheeted surfaces $S_1(n_1, \ldots, n_{k-1})$ carried by $\Sigma[b_1, \ldots, b_k]$ are orientable if and only if each b_i is even. There is only one such continued fraction expansion $p/q = r + [b_1, \ldots, b_k]$ with each b_i even, so we deduce:

Corollary. *The orientable incompressible Seifert surfaces for $K_{p/q}$ all have the same genus, and are all isotopic if and only if at most one of the b_i's in the unique expansion $p/q = r + [b_1, \ldots, b_k]$ with all b_i's even is not ± 2.*

[The case that all b_i's are ± 2 is the case that $S^3 - K_{p/q}$ fibers over a circle, the fiber being the unique incompressible orientable Seifert surface.]

Proposition 2. *Each boundary circle of a surface $S_n(n_1, \ldots, n_{k-1})$ carried by $\Sigma[b_1, \ldots, b_k]$ wraps around $K_{p/q}$ once longitudinally and $m(b_1, \ldots, b_k)$ times meridionally, $m(b_1, \ldots, b_k)$ being the function*

$$m(b_1, \ldots, b_k) = 2[(n^+ - n^-) - (n_0^+ - n_0^-)],$$

where n^+ and n^- are the number of positive and negative b_i's, and n_0^+ and n_0^- are the corresponding numbers for the unique continued fraction expansion $p/q = r' + [b'_1, \ldots, b'_n]$ with each b'_i even.

Examples

p/q	Minimal $[b_1, \dots, b_k]$'s	Corresp. values of $2[(n^+ - n^-) - (n_0^+ - n_0^-)]$	Number of connected incompr. n-sheeted surfaces, $n = 1, 2, \dots$
$1/q$	$\underbrace{[-2, -2, \dots, -2]}_{q-1}, [q]$	$0, 2q$	$2, 1, 0, 0, \dots$
$3/5$	$[-2, 2], [2, 3], [-3, -2]$	$0, 4, -4$	$3, 2, 0, 0, \dots$
$3/7$	$[-2, -4], [2, -3], [3, 2, 2]$	$0, 4, 10$	$3, 2, 0, 0, \dots$
$4/11$	$[2, -2, -2, -2], [3, 4], [-2, -3, -2, -2], [-2, -2, -3]$	$0, 8, -4, 2$	$5, 5, 0, 0, \dots$
$5/13$	$[-2, -2, 2, 2], [-2, -3, -3], [2, -2, -3], [3, 3, 2], [3, 2, -2]$	$0, -6, -2, 6, 2$	$7, 8, 4, 4, 8, 8, 12, 8$

Note. $K_{3/5}$ and $K_{5/13}$ are amphicheiral (the condition is $p^2 \equiv -1 \pmod{q}$) which accounts for the symmetry in their data.

The result of the Dehn surgery on $K_{p/q}$ in which a tubular neighborhood of $K_{p/q}$ is cut out and reglued in so as to make a meridian disc kill a curve in $S^3 - K_{p/q}$ wrapping l times around $K_{p/q}$ longitudinally and m times meridionally, we shall call $M_{m/l}(K_{p/q})$.

Theorem 2. (a) *$M_{m/l}(K_{p/q})$ is irreducible, with the trivial exceptions*

(i) *$p \equiv \pm 1 \pmod{q}$ and $m/l = \pm 2q$, in which case*

$$M_{\pm 2q}(K_{\pm 1/q}) = L_{p,q} \# L_{q,p}.$$

(ii) *$p \equiv 0 \pmod{q}$ and $m/l = 0$, in which case $M_0(K_0) = S^1 \times S^2$.*

(b) *If $M_{m/l}(K_{p/q})$ is a Haken manifold, then $l = 1$ and $m = m(b_1, \dots, b_k)$ for one of the finitely many continued fraction expansions $p/q = r + [b_1, \dots, b_k]$ satisfying $|b_i| \geqq 2$ for each i.*

J. Przytycki [13] has proved the converse of (b), for p/q not one of the exceptions in (a).

Since a 2-bridge knot $K_{p/q}$ is simple, i.e., every incompressible torus in $S^3 - K_{p/q}$ is isotopic to the peripheral torus, then except for the torus knots (when $S^3 - K_{p/q}$ is Seifert-fibered), the main result of [19] asserts that $S^3 - K_{p/q}$ has a hyperbolic structure which is complete and of finite volume. Hence all but finitely many Dehn surgeries on $K_{p/q}$ yield hyperbolic manifolds [18]. Being hyperbolic, these are not Seifert-fibered.

Theorem 3. *Any diffeomorphism of 2-bridge knot complements $S^3-K_{p/q}\to S^3-K_{r/s}$ can be isotoped so that it extends to a diffeomorphism of S^3. In particular, $K_{p/q}$ is isotopic to $K_{r/s}$, or its mirror image, if $K_{p/q}$ and $K_{r/s}$ have diffeomorphic complements.*

The proof of this is easy from Theorem 1 and Proposition 2, so we give the argument here. For a knot $K\subset S^3$, let $\mathcal{S}(K)\subset\mathbb{Q}\cup\{1/0\}$ be the set of slopes of boundary curves of incompressible, ∂-incompressible surfaces in S^3-K. According to [8], $\mathcal{S}(K)$ is finite, and by [4], $\mathcal{S}(K)$ has at least two elements (including 0, of course).

Lemma 1. *Suppose* (a) $\mathcal{S}(K)\subset\mathbb{Z}\cup\{1/0\}$ *and* (b) *there exist* $m_1, m_2\in\mathcal{S}(K)-\{1/0\}$ *with* $|m_1-m_2|>2$. *Then a meridian circle of* K *is determined (up to isotopy) solely by* S^3-K.

Proof. Condition (a) implies that, in the Diagram (Fig. 4), the vertex 1/0 is joined by an edge to each slope in $\mathcal{S}(K)-\{1/0\}$, and condition (b) implies that 1/0 is the only vertex with this property. Thus the slope 1/0 is uniquely determined by $\mathcal{S}(K)$, hence also by S^3-K. □

Theorem 3 follows from Lemma 1 since (a) and (b) are satisfied for 2-bridge knots. (For (b), consider the two edge-paths which form the top and bottom borders of the strip in Fig. 5. For one of these edge-paths $n^+=0$, while $n^-=0$ for the other. So by Proposition 2 their slopes m_1 and m_2 satisfy $|m_1-m_2|>2$.)

Question. Do (a) and (b) of Lemma 1 hold for all nontrivial knots?

§ 2. Proof of Theorem 1

We shall use the picture of $K=K_{p/q}$ in Fig. 1(b), but with the height function regarded as the natural projection $S^3\to\mathbb{R}$, making the levels 2-spheres $S^2_r=S^2\times\{r\}\subset S^2\times\mathbb{R}\subset S^3$. Each level S^2_r we identify with the orbit space $\mathbb{R}^2/\Gamma$ where Γ is the group generated by 180° rotations of $\mathbb{R}^2$ about the integer lattice points. Let $\mathring{S}^2_r$ be $S^2_r-\mathbb{Z}^2/\Gamma$, a 4-punctured sphere. We take as known the elementary facts:

(i) The isotopy classes of smooth circles in $\mathring{S}^2_r$ separating the four punctures into pairs are in one-to-one correspondence with $\mathbb{Q}\cup\{1/0\}$.

(ii) The isotopy classes of smooth arcs in $\mathring{S}^0_r$ joining one given puncture to any of the other three punctures are in one-to-one correspondence with $\mathbb{Q}\cup\{1/0\}$.

In either case, a representative of the isotopy class corresponding to $a/b\in\mathbb{Q}\cup\{1/0\}$ is the projection to $S^2=\mathbb{R}^2/\Gamma$ of a line in $\mathbb{R}^2$ of slope a/b (the line being disjoint from $\mathbb{Z}^2$ in (i), and intersecting $\mathbb{Z}^2$ in (ii)). The number $a/b\in\mathbb{Q}\cup\{1/0\}$ associated to a circle or arc in (i) or (ii) is called its *slope*. We may assume $K\subset S^2\times[0,1]$, with $K\cap S^2_r$ consisting of two arcs of slope 1/0 for $r=1$, two arcs of slope p/q for $r=0$, and the four points of $\mathbb{Z}^2/\Gamma$ for $0<r<1$.

Proof of Theorem 1(a). Let S be a closed incompressible surface in S^3-K. We may suppose the height function on S is a morse function. For each non-critical level between the top and bottom of K, we define the slope of this level to be the slope of any circles of S in this level which separate the four points of K into pairs, if there are any such circles (if there are several such circles, they must all have the same slope). Near the top of K the slope, if defined, is $1/0$, and near the bottom of K it is p/q, if defined. Passing between these two extreme levels, the slope can change only at the level of a saddle of S. At a saddle, either one level circle of S splits into two level circles, or two level circles are joined into one level circle. In either case, the three level circles, after projecting to a common level S^2, can be isotoped in S^2-K to be disjoint. Disjoint circles cannot have different slopes, so the slope cannot change at a saddle, except to become undefined.

Assuming $p/q \neq 1/0$, there must therefore be some non-critical level S^2 between the top and bottom of K where the slope is undefined. In either ball B^3 bounded by this level S_r^2, K consists of two standard arcs (up to diffeomorphism).

Let $D^2 \subset B^3$ be a disc separating the two arcs of K, chosen so that $\partial D^2 \subset S_r^2 - S$. This is possible since none of the circles of $S \cap S_r^2$ separate the four points of $S_r^2 \cap K$ into pairs. By the incompressibility of S, we may isotope S to be disjoint from D^2. (Consider an innermost circle of $S \cap D^2$. The disc it bounds on S may be replaced by the disc it bounds in D^2, by an isotopy of S since the union of these two discs is a sphere in S^3-K, which bounds a ball since S^3-K is irreducible.) So there are tubes about the two arcs of K in B^3 with $S \cap B^3$ contained in these two tubes. And similarly for the other 3-ball bounded by S_r^2. These four tubes can be chosen to fit together to form a tubular neighborhood N of K, with $S \subset N$. S is incompressible in N, so by a standard argument, S must be a torus isotopic to ∂N. □

Let S be an incompressible, ∂-incompressible surface in $S^3-\mathring{N}$, N a tubular neighborhood of K, $\partial S \neq \emptyset$. The components of ∂S are non-trivial circles on the torus ∂N, each wrapping around m times longitudinally and l times meridionally, say. The possibility $l=0$ can be ruled out by the same argument which proved Theorem 1(a). Hence we may *assume that ∂S intersects each meridian circle of ∂N transversely.* If we regard N as being very small, infinitesimally small in fact, then the surface S near K consists of a fixed number n of sheets meeting along K.

The height function on S we may assume to be a morse function, with critical points all in distinct levels. In a given non-critical level S_r^2 between the top of K and the bottom, $S \cap S_r^2$ consists of a finite number of circles and arcs joining the four points of $S_r^2 \cap K$ in some pattern, n arcs meeting at each point of $S_r^2 \cap K$. If any arc of $S \cap S_r^2$ joins one point of $S_r^2 \cap K$ to itself, then there must be such an arc bounding a disc in $S_r^2 - K$. Then by the ∂-incompressibility of S, ∂S could not be transverse to all the meridian circles of ∂N. Hence we may *assume each non-critical level arc of S has distinct endpoints on K.*

Any circles of $S \cap S_r^2$ must either be trivial, bounding discs in $S_r^2 - K$, or they must separate the four points of $S_r^2 \cap K$ into pairs and have the same slope as the arcs of $S \cap S_r^2$. If there are non-trivial circles in $S \cap S_r^2$, then after an isotopy of $S_r^2 - K$ and a linear change of coordinates (by the action of $GL(2, \mathbb{Z})$ on $S_r^2 = \mathbb{R}^2/\Gamma$), and ignoring trivial circles, $S \cap S_r^2$ can be put in the form

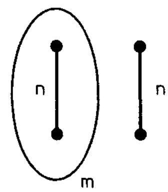
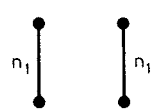


Fig. 6

where the labels m and n indicate the numbers of parallel copies of the curves they are affixed to.

Consider now the case that $S \cap S_r^2$ contains no non-trivial circles. By a simple counting argument, one sees that if there are n_1 arcs joining one pair of points of $S_r^2 \cap K$, then there must be n_1 arcs joining the opposite pair also. By isotopy and linear change of coordinates, and ignoring trivial circles, these $2n_1$ arcs can be put in a standard position:



If $n_1 \neq n$ there will be some arcs joining the two halves of this picture. By further isotopy and linear change of coordinates the picture can be enlarged to:

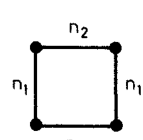


Fig. 7

There can be no arcs of a third slope. For if there were the picture would become:

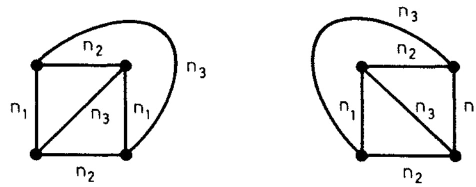


Fig. 8

As the given level S_r^2 is varied, this configuration must eventually change to one with different n_i's. This can only occur in passing a saddle of S which joins two level arcs of S having a common endpoint, e.g.,

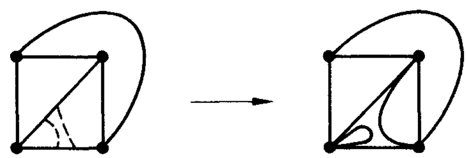


Fig. 9

This creates a level arc joining a point of $S_r^2 \cap K$ to itself, violating our hypotheses on S. Thus we have shown that in a transverse level S_r^2, $S \cap S_r^2$ can contain arcs of at most two slopes. These slopes a/b and c/d satisfy $ad-bc=\pm 1$, since a linear change of coordinates brought them to slopes 0 and ∞.

For the given level S_r^2 we define $\lambda(S_r^2)$ to be the point in the Diagram (of Fig. 4) having barycentric coordinates $\frac{m}{n}\left\langle\frac{a}{b}\right\rangle+\frac{n-m}{n}\left\langle\frac{c}{d}\right\rangle$, where there are $2m$ arcs of $S \cap S_r^2$ of slope a/b and $2n-2m$ arcs of slope c/d. Near the top of K, $\lambda(S_r^2)=1/0$, and near the bottom, $\lambda(S_r^2)=p/q$. In between, $\lambda(S_r^2)$ can change only at a saddle of S. A saddle at which $\lambda(S_r^2)$ does in fact change we call an *essential saddle*. Thus an essential saddle joins two level arcs whose four endpoints are the four strands of K. Two successive $\lambda(S_r^2)$'s lie in a common edge of the Diagram and have barycentric coordinates of the form

$$\frac{m}{n}\left\langle\frac{a}{b}\right\rangle+\frac{n-m}{n}\left\langle\frac{c}{d}\right\rangle \quad \text{and} \quad \frac{m+1}{n}\left\langle\frac{a}{b}\right\rangle+\frac{n-m-1}{n}\left\langle\frac{c}{d}\right\rangle.$$

Let $\lambda_0=1/0, \lambda_1, \ldots, \lambda_l=p/q$ be the sequence of $\lambda(S_r^2)$'s from the top of K to the bottom, $\lambda_i \neq \lambda_{i-1}$.

Lemma 2. *S, if incompressible and ∂-incompressible, can be isotoped* (rel K) *so as to eliminate all its critical points which are not essential saddles. The λ-sequence of this new S satisfies:*

(i) *no three successive λ_i's lie on two different edges of a triangle of the Diagram;*

(ii) *$\lambda_i \neq \lambda_{i+2}$ for each i.*

Proof. If there are any trivial circles of $S \cap S_r^2$ for $r \in [0,1]$, then S can be isotoped to produce an index 0 or 2 critical point in a level S_r^2, $0<r<1$, and the procedure of Proposition 2.1 of [5] can be applied to eliminate such critical points. So we may assume $S \cap S_r^2$ has no trivial circles in levels S_r^2 meeting K.

Next we show how to eliminate also nontrivial circles of $S \cap S_r^2$ for $0 \leqq r \leqq 1$, and in particular make S disjoint from levels S_r^2 not meeting K. Nontrivial circles of $S \cap S_r^2$ arise either from circles of $S \cap S_0^2$ and $S \cap S_1^2$, or from saddles of S in levels S_r^2, $0 < r < 1$, joining an arc of $S \cap S_r^2$ to itself:

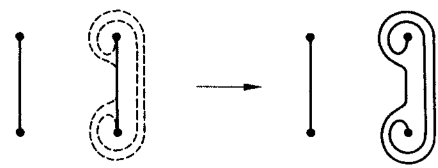


Fig. 10

Consider such a saddle which is followed by another saddle which decreases the number of nontrivial level circles of S. Examining the various possible positions for such a second saddle, one finds that the second saddle can be deformed to lie in the same level as the first saddle:

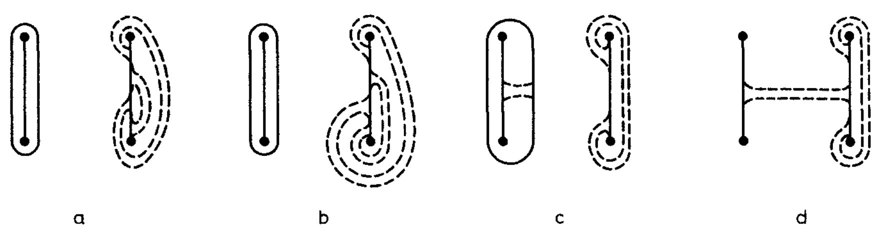


Fig. 11

Then one considers what happens if the heights of the two saddles are interchanged. In (a), a trivial level circle is produced, a situation we have already seen how to simplify. In (b), S is obviously ∂-compressible. In (c) and (d), interchanging the heights of the two saddles decreases the number of level circles. Thus we may always eliminate saddles joining a level arc of S to itself. Any circles of $S \cap S_0^2$ or $S \cap S_1^2$ must be joined to level arcs of S by saddles. Such circle-saddle pairs can easily be cancelled by "pushing across the point at ∞":

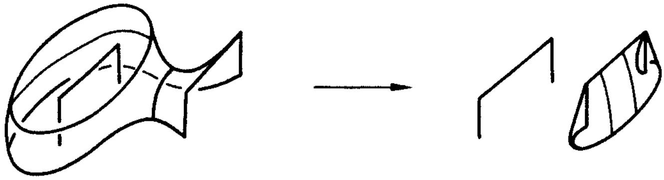


Fig. 12

Thus we have eliminated all level circles of S, and so the only critical points remaining on S are essential saddles.

It remains to show the λ-sequence of S satisfies (i) and (ii).

(i) If two successive saddles give λ_i's on adjacent sides of a triangle, then (after a change of coordinates) they have the form

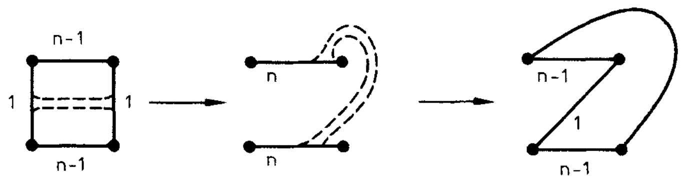


Fig. 13

These two saddles can be put on the same level:

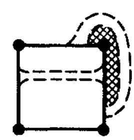


Fig. 14

The shaded disc then exhibits S as ∂-compressible.

(ii) There are just two possibilities for successive saddles yielding $\lambda_i = \lambda_{i+2}$ (up to change of coordinates):

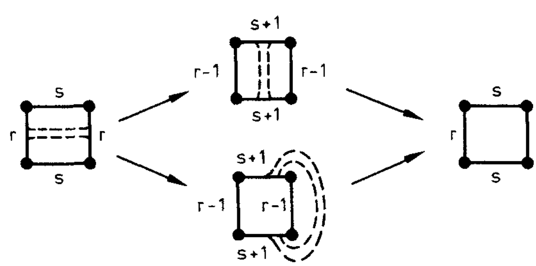


Fig. 15

In the first of these two sequences, S is clearly compressible. The second sequence is also clearly compressible if $i=0$ (hence $s=0$). So we assume $i>0$. In the second sequence above, the two saddles can be put on the same level. If $s>0$, then reversing the order of the two saddles yields a new λ-sequence with $\lambda_{i+1} = \lambda_{i-1}$:

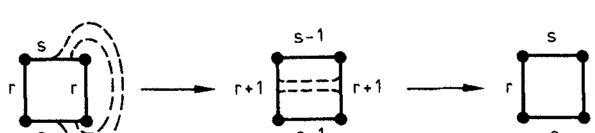


Fig. 16

So in this case we are done by induction on i. If $s=0$, then after reversing the order of the two saddles, one saddle can be slid over the other, changing the configuration from

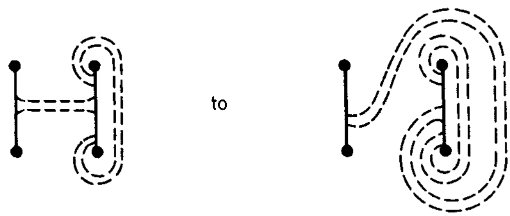


Fig. 17

As a result, the sequence $\lambda_i, \lambda_{i+1}, \lambda_{i+2}$ can be changed to $\lambda_i, \lambda'_{i+1}, \lambda_{i+2}$ where the edge of the Diagram containing λ_i and λ'_{i+1} is two triangles away from the edge containing λ_i and λ_{i+1}:

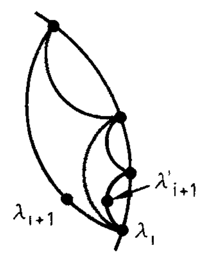


Fig. 18

This trick can be repeated until the edge containing λ_i and λ_{i+1} is either the same as, or one edge removed from, the edge containing λ_{i-1} and λ_i. The latter possibility is ruled out by (i), while in the former case there results a λ-sequence with $\lambda_{i+1}=\lambda_{i-1}$. So by induction on i we have established (ii). □

Proof of Theorem 1(b). Let us fix the number of sheets, n. Clearly, λ-sequences satisfying (i) and (ii) then correspond bijectively with minimal edge-paths from 1/0 to p/q, hence also with continued fraction expansions $p/q=r+[b_1, \ldots, b_k]$ with $|b_i|\geq 2$. Fixing one such expression for p/q, let $p_i/q_i=r+[b_1, \ldots, b_i]$. The branched surface $\Sigma[b_1, \ldots, b_k]$, when repositioned so that $\partial\Sigma[b_1, \ldots, b_k]$ is the $K_{p/q}$ of Fig. 1b, has the properties:

(1) In levels between the i^{th} and $i+1^{\text{st}}$ pairs of complementary horizontal plumbing squares (actually, parallelograms), $\Sigma[b_1, \ldots, b_k]$ consists of two arcs of slope p_i/q_i, for $1\leq i\leq k-2$.

(2) At the top two arcs of K, $\Sigma[b_1, \ldots, b_k]$ has slope 1/0, and just below the saddle of $\Sigma[b_1, \ldots, b_k]$ near the top of K, $\Sigma[b_1, \ldots, b_k]$ has slope r.

(3) At the bottom two arcs of K, $\Sigma[b_1, \ldots, b_k]$ has slope p/q, and just above the saddle of $\Sigma[b_1, \ldots, b_k]$ near the bottom of K, $\Sigma[b_1, \ldots, b_k]$ has slope p_{k-1}/q_{k-1}.

A surface $S_n(n_1, \ldots, n_{k-1})$ carried by $\Sigma[b_1, \ldots, b_k]$ has exactly $n(k+1)$ saddles, all essential, and the associated λ-sequence is the one corresponding to the expansion $p/q=r+[b_1, \ldots, b_k]$. Conversely, any n-sheeted surface S with this λ-sequence and only essential saddles is isotopic to such an $S_n(n_1, \ldots, n_{k-1})$, for some choice of $n_1, \ldots, n_{k-1}$. For, the top n saddles of S have a unique position, up to isotopy, since the two choices for the first saddle

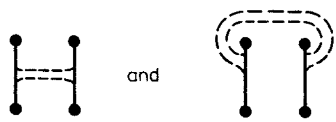
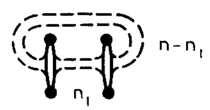


Fig. 19

yield isotopic surfaces. Similarly for the bottom n saddles. For the i^{th} intermediate bunch of n saddles there are $n+1$ possible arrangements, with n_i "inner" saddles and $n-n_i$ "outer" saddles:



□

Fig. 20

Proof of Theorem 1(c). The "only if" statement follows from Lemma 2 and the preceding remarks. For the "if" half we shall verify that $\Sigma=\Sigma[b_1, \ldots, b_k]$ satisfies the conditions of [7] and [12] which imply that any surface carried by Σ is incompressible and ∂-incompressible. Namely:

(1) Σ carries some surface with strictly positive weights, for example the surface $S_2(1, \ldots, 1)$.

(2) Σ has no Reeb branched subsurfaces, since it carries no tori (obviously) nor ∂-compressible annuli (since all surfaces carried by Σ have negative Euler characteristic except in the trivial case $k=1$).

(3) Σ has no disks of contact since there are no circles in the branching locus of Σ. Also, Σ has no half-disks of contact since the branching arcs of Σ are nontrivial in $H_1(\Sigma, \partial\Sigma; \mathbb{Z}_2)$.

(4) There are no monogons in $(S^3-K)-\Sigma$ with boundary on Σ. To see this, consider the complementary component V_i of Σ in S^3-K which meets the i^{th} twisted band in Σ (see Figs. 2 and 3). Topologically, V_i is a solid torus, containing in its boundary one or two circles (depending on whether b_i is odd or even) formed by arcs of K and arcs C of "cusp points" coming from the branching locus of Σ. Observe that $K\cup C$ meets a meridian disk of V_i in at least $|b_i|$ points.

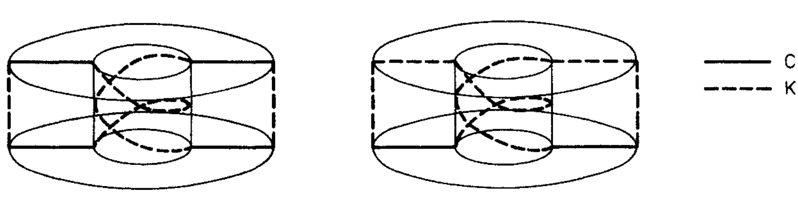


Fig. 21

Since $|b_i| \geqq 2$, this rules out a meridian disk of V_i being a monogon. (The monogon is disjoint from K by hypothesis.) Also, a trivial circle of ∂V_i meets $K \cup C$ transversely in an even number of points, so cannot bound a monogon either.

(5) The final condition from [7], [12] is that the horizontal boundary $\partial_h N$ of a fibered neighborhood N of Σ be incompressible and ∂-incompressible in $(S^3 - K) - N$. In the present case, the part of $\partial_h N$ in V_i is $\partial V_i - (K \cup C)$, which consists of one or two annuli in ∂V_i. These are incompressible and ∂-incompressible in V_i if $|b_i| \geqq 2$. $\square$

Proof of Theorem 1(d). Consider an isotopy S_t from one surface $S_n(n_1, \ldots, n_k)$ to another such surface. Generically, the height function on S_t will have only nondegenerate critical points, all on distinct levels, except for the following isolated phenomena:

(A) A pair of nondegenerate critical points of adjacent indices is introduced or cancelled in a level not near other critical levels.

(B) Two nondegenerate critical points interchange levels.

The λ-sequence of S_t is defined for S_t not containing type (A) or (B) phenomena, by ignoring level circles. To prove Theorem 1(d) we must see that phenomena (A) and (B) do not change the λ-sequence. Since (A) and (B) involve only two critical points, at most one λ_i in the λ-sequence can be affected. For a λ-sequence satisfying (i) and (ii) of Lemma 2, each λ_i is determined by λ_{i-1} and λ_{i+1}, at least if n, the number of sheets of S_t, is greater than 1. But for proving Theorem 1(d) we are free to replace the given surface $S_n(n_1, \ldots, n_k)$ by $S_{2n}(2n_1, \ldots, 2n_k)$, the boundary of a tubular neighborhood of $S_n(n_1, \ldots, n_k)$. $\square$

Proof of Theorem 1e. We have seen that the λ-sequence of an incompressible, ∂-incompressible $S \subset S^3 - K$ determines S up to isotopy, and up to a 2-fold ambiguity for each essential saddle. We have to examine how an isotopy of S could reverse the type of an essential saddle. For a generic isotopy, the only thing which could reverse the type of an essential saddle is interchanging the relative heights of this saddle and another saddle. If the type of the saddle is reversed, then the two saddles, when put on the same level, have to attach to four different sides of level arcs, e.g.,

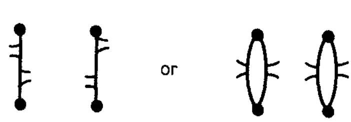


Fig. 22

Then both saddles are essential, and it is easy to see that when the two saddles are put on the same level, the only possible configurations, up to level-preserving isotopy and linear change of coordinates, is:

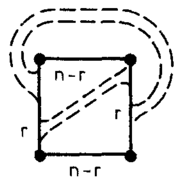


Fig. 23

Here the inner saddle is the first one, changing λ_i to λ_{i+1}, and the outer saddle changes λ_{i+1} to λ_{i+2}. If $r>1$, then the two saddles represent, in terms of Fig. 3, a complementary pair of horizontal plumbing squares. Hence the same surface $S_n(n_1, \ldots, n_{k-1})$ results from interchanging the heights of the two saddles. Also, λ_i, λ_{i+1}, and λ_{i+2} are unchanged. If $r=1$ (so λ_{i+1} is a vertex of the Diagram), then interchanging the heights of the two saddles changes

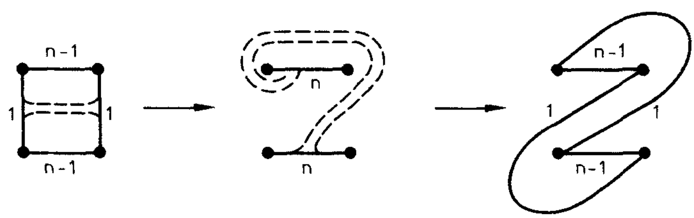


Fig. 24

to

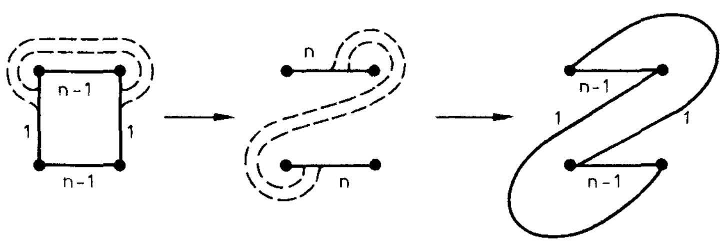


Fig. 25

The λ-sequence is unchanged, but $S_n(n_1, \ldots, n_{k-1})$ is isotoped to $S_n(n_1, \ldots, n_j \pm 1, n_{j+1} \pm 1, \ldots, n_{k-1})$ for some j, $0 \leqq j \leqq k-1$, where $b_j = \pm 2$ since the slopes involved in these configurations are 1/0, 0/1, and 1/2. □

§ 3. Proof of Proposition 2

Let l and m denote the longitudinal and meridional wrapping numbers of the given surface $S_n(n_1, \ldots, n_{k-1})$. We first show $l=1$. To do this, we are free to change any b_i by an even integer, since this just adds some number of full twists to a vertical portion of $\Sigma[b_1, \ldots, b_k]$. So we may assume the level arcs of our surface $S_n(n_1, \ldots, n_{k-1})$ have slopes 0/1, 1/1, or 1/0 (obtained by reducing mod two the numerators and denominators of the slopes of level arcs of the original surface). Thus after each bunch of n saddles, the level set can be normalized by isotopy to be one of:

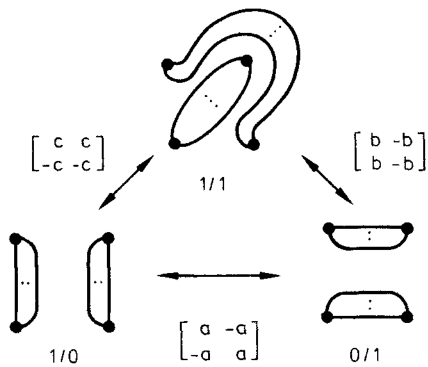


Fig. 26

Going from one of these level sets to another via n saddles, the n sheets meeting at each vertex are rotated by some number of notches, indicated by the 2×2 matrices.

Example.

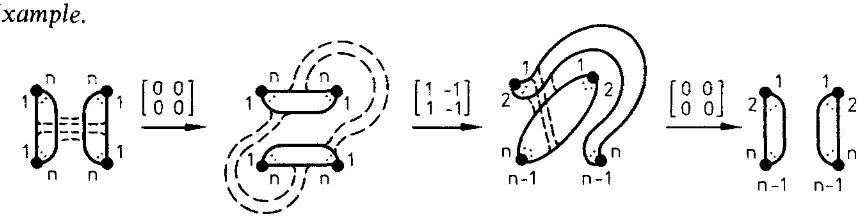


Fig. 27

Since $K_{p/q}$ is a knot rather than a link, the final level set has mod 2 slope 0/1 or 1/1. We may assume the final mod 2 slope is 0/1, in fact, since $K_{p/q} = K_{(p+q)/q}$. Thus the total effect of all saddles, mod 2, is a change in level sets:

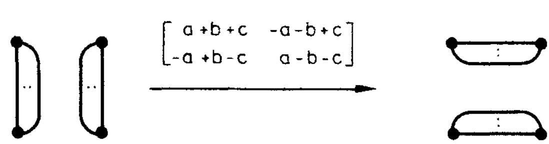


Fig. 28

Note that a and b do not affect the way in which sheets are identified in the slope 0/1 picture, while the $\pm c$ rotations affect these identifications in the top and bottom halves of the picture in the same way. So, by following a sheet all the way around K one returns to the same sheet, hence $l=1$.

Next we show that m depends only on $[b_1, \ldots, b_k]$. Consider first the single-sheeted case $S_1(n_1, \ldots, n_{k-1})$. Changing some n_i changes the surface near K by adding a positive full twist around two of the four strands of K in Fig. 3 and a negative full twist in the other two strands. So the number m is unchanged. (The sense of a twist does not depend on an orientation of K, just an orientation of S^3.) Similarly, in the n-sheeted case the sum of the m's for all n boundary circles is independent of the n_i's. Since all these n boundary circles have equal m's, m for each one is also independent of the n_i's. Clearly, m for the surfaces $S_n(0, \ldots, 0)$ and $S_1(0, \ldots, 0)$ is the same, so m is also independent of n.

Finally, we compute the function $m = m(b_1, \ldots, b_k)$. Without loss of generality we may assume $p/q > 0$ and restrict to edge-paths lying in the upper half of the Diagram. For such an edge-path, define $N^+(N^-)$ to be the number of edges of the edge-path where the slope increases (decreases), the slope 1/0 being regarded as $+\infty$. Note that $N^+ = n^+$ and $N^- = n^- + 1$. The quantity $2(N^+ - N^-) - m$ depends only on the endpoints of the edge-path, we claim. To see this, it suffices to check that $2(N^+ - N^-) - m$ vanishes for the closed edge-paths

$$\langle 1/0 \rangle \to \langle 0/1 \rangle \to \langle 1/0 \rangle \quad \text{and} \quad \langle 1/0 \rangle \to \langle 0/1 \rangle \to \langle 1/1 \rangle \to \langle 1/0 \rangle .$$

This is obvious in the first case, while in the second case one has the pictures

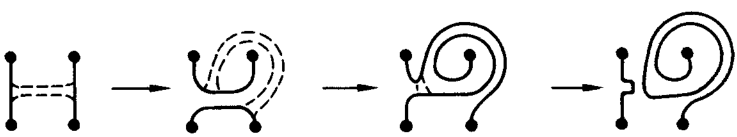


Fig. 29

Straightening out the last picture to look like the first one requires two full turns at the upper right strand. So $2(N^+ - N^-) - m = 2(2-1) - 2 = 0$.

For the unique minimal edge-path from 1/0 to p/q corresponding to an oriented single-sheeted incompressible surface (i.e., all b_i's even), $2(N^+ - N^-) - m$ is by definition $2(N_0^+ - N_0^-) - 0$. Hence for any edge-path from 1/0 to p/q (in the upper half of the Diagram), $2(N^+ - N^-) - m = 2(N_0^+ - N_0^-)$, or $m = 2[(N^+ - N^-) - (N_0^+ - N_0^-)]$, which equals $2[(n^+ - n^-) - (n_0^+ - n_0^-)]$. $\square$

§4. Dehn Surgery

For the manifold $M_{m/l}(K)$ obtained by m/l Dehn surgery on an arbitrary knot $K \subset S^3$, there is the following basic result:

Lemma 3. (a) *If $M_{m/l}(K)$ is not irreducible, then there is an incompressible, ∂-incompressible genus zero surface in $S^3 - \mathring{N}(K)$ whose boundary circles have slope m/l in $\partial N(K)$.*

(b) *If $M_{m/l}(K)$ is irreducible and contains an orientable incompressible surface, then $S^3 - \mathring{N}(K)$ contains an orientable incompressible surface which is either closed and not a torus isotopic to $\partial N(K)$, or bounded and ∂-incompressible with boundary circles of slope m/l in $\partial N(K)$.*

Proof. (a) Let $S \subset M_{m/l}(K)$ be a 2-sphere not bounding a 3-ball. By transversality, we may assume S intersects $N(K) \subset M_{m/l}(K)$ in a number of meridian discs. This number cannot be zero since $S^3 - K$ is irreducible. If $S \cap (S^3 - \mathring{N}(K))$ is not incompressible in $S^3 - \mathring{N}(K)$, then surgering S along a compressing disc yields two 2-spheres in $M_{m/l}(K)$, each intersecting $N(K)$ in fewer meridian discs than S. If both these 2-spheres bound balls in $M_{m/l}(K)$, so would the original S. So by minimizing the number of meridian discs of $S \cap N(K)$ we obtain a 2-sphere $S \subset M_{m/l}(K)$ not bounding a ball in $M_{m/l}(K)$, such that $S \cap (S^3 - \mathring{N}(K))$ is incompressible in $S^2 - \mathring{N}(K)$. This genus zero surface in $S^3 - \mathring{N}(K)$ must also be ∂-incompressible, since otherwise it would be a ∂-parallel annulus (as mentioned at the beginning of § 1), and then S could be pushed into $N(K)$, where it would bound a ball.

(b) Let $S \subset M_{m/l}(K)$ be an orientable incompressible surface, intersecting (we may assume) $N(K)$ in a number of meridian discs. Surgering S across a compressing disc for $S \cap (S^3 - \mathring{N}(K))$ in $S^3 - \mathring{N}(K)$, if any exists, must split S into a 2-sphere S^2 and another surface S', since S is incompressible in $M_{m/l}(K)$. This S^2 bounds a 3-ball in $M_{m/l}(K)$ since $M_{m/l}(K)$ is irreducible. Hence S is isotopic to S'. Moreover, S' meets $N(K)$ in fewer meridian discs than S, since $S^2 \cap N(K) \neq \emptyset$. This process of isotoping S to eliminate intersections with $N(K)$ can be repeated until $S \cap (S^3 - \mathring{N}(K))$ is incompressible in $S^3 - \mathring{N}(K)$. It must also then be ∂-incompressible, otherwise it would be a ∂-parallel annulus and S would be a 2-sphere. □

Proof of Theorem 2. (a) All the single-sheeted surfaces carried by a given $\Sigma[b_1, \ldots, b_k]$ are diffeomorphic, and any n-sheeted surface $S_n = S_n(n_1, \ldots, n_{k-1})$ carried by $\Sigma[b_1, \ldots, b_k]$ is, abstractly, an n-fold covering space of this single-sheeted S_1. Hence $\chi(S_n) = n\chi(S_1) = n(1-k)$. Let $\hat{S}_n$ be obtained from S_n by capping off its n (by Proposition 2) boundary circles with discs. If S_n is orientable, connected, and of genus g, then $2 - 2g = \chi(\hat{S}_n) = n(1-k) + n = n(2-k)$. So $g = 0$ only when $(k, n) = (0, 1)$ or $(1, 2)$. If $k = 0$, $K_{p/q}$ is the trivial knot, yielding the exception (ii) in Theorem 2. In the case $k = 1$, $p \equiv \pm 1 \pmod{q}$ and K is a torus knot. Lemma 3 and the formula for $m(b_1, \ldots, b_k)$ then yield the exceptions in (i).

(b) This is immediate from Proposition 2 and Lemma 3. □

§ 5. Proof of Proposition 1

For this, only the mod 2 values of the b_i's are relevant, so we can assume the edge-path involves only the slopes 1/0, 0/1, and 1/1, just as at the beginning of the proof of Proposition 2.

Consider three successive vertices of the mod 2 edge-path. There are two possibilities (up to change of coordinates), according to whether the b_i in question is even or odd:

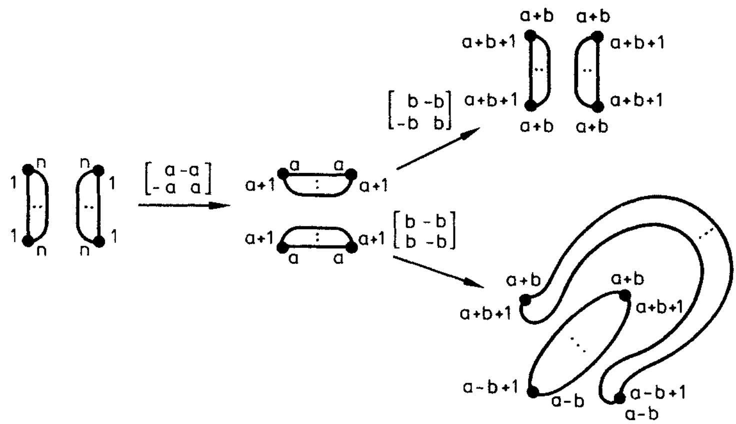


Fig. 30

After the first bunch of n saddles, the four cycles of n sheets at the four strands of K are identified to one cycle. In the case when the third slope is the same as the first (b_i even), there is no further identification of sheets in the third slope's level set. In particular, if all b_i's are even, the surface has n components.

In the other case when the third slope is different from the first (b_i odd), the cycle of n sheets is folded in half along the diameter separating a from $a+1$ (independent of what b is):

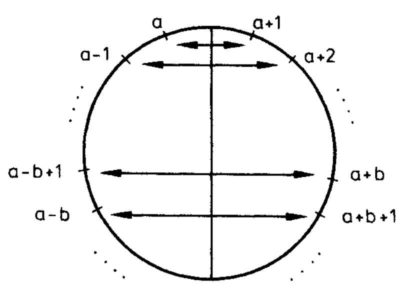


Fig. 31

So if some b_i is odd, a two-sheeted surface is connected. If only one b_i is odd, there is just one folding, so the n sheets are identified at most two-to-one, hence a connected surface can have at most two sheets. If at least two b_i's are odd, there are two foldings of the n-cycle. These can be chosen arbitrarily, by suitable choice of "a" at each bunch of n saddles, so one can realize a rotation by two notches of the n-cycle of sheets. If n is odd, this already means all sheets are connected together. If n is even, it means alternate sheets are connected, but then a single folding already connected two adjacent sheets. (E.g., the Example given in the proof of Proposition 2 is a connected n-sheeted surface.) □

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Oblatum 1-III & 16-V-1983