Lemma 3.27. *Let M be a manifold of dimension n and let $A \subset M$ be a compact subset. Then:*

(a) *If $x \mapsto \alpha_x$ is a section of the covering space $M_R \to M$, then there is a unique class $\alpha_A \in H_n(M|A;R)$ whose image in $H_n(M|x;R)$ is α_x for all $x \in A$.*

(b) *$H_i(M|A;R) = 0$ for $i > n$.*

To deduce the theorem from this, choose $A = M$, a compact set by assumption. Part (c) of the theorem is immediate from (b) of the lemma. To obtain (a) and (b) of the theorem, let $\Gamma_R(M)$ be the set of sections of $M_R \to M$. The sum of two sections is a section, and a scalar multiple of a section is a section, so $\Gamma_R(M)$ is an R-module. There is a homomorphism $H_n(M;R) \to \Gamma_R(M)$ sending a class α to the section $x \mapsto \alpha_x$, where α_x is the image of α under the map $H_n(M;R) \to H_n(M|x;R)$. Part (a) of the lemma asserts that this homomorphism is an isomorphism. If M is connected, each section is uniquely determined by its value at one point, so statements (a) and (b) of the theorem are apparent from the earlier discussion of the structure of M_R. □

Proof of 3.27: The coefficient ring R will play no special role in the argument so we shall omit it from the notation. We break the proof up into four steps.

(1) First we observe that if the lemma is true for compact sets A, B, and $A \cap B$, then it is true for $A \cup B$. To see this, consider the Mayer–Vietoris sequence

$$0 \longrightarrow H_n(M|A \cup B) \xrightarrow{\Phi} H_n(M|A) \oplus H_n(M|B) \xrightarrow{\Psi} H_n(M|A \cap B)$$

Here the zero on the left comes from the assumption that $H_{n+1}(M|A \cap B) = 0$. The map Φ is $\Phi(\alpha) = (\alpha, -\alpha)$ and Ψ is $\Psi(\alpha, \beta) = \alpha + \beta$, where we omit notation for maps on homology induced by inclusion. The terms $H_i(M|A \cup B)$ farther to the left in this sequence are sandwiched between groups that are zero by assumption, so $H_i(M|A \cup B) = 0$ for $i > n$. This gives (b). For the existence half of (a), if $x \mapsto \alpha_x$ is a section, the hypothesis gives unique classes $\alpha_A \in H_n(M|A)$, $\alpha_B \in H_n(M|B)$, and $\alpha_{A\cap B} \in H_n(M|A \cap B)$ having image α_x for all x in A, B, or $A \cap B$ respectively. The images of α_A and α_B in $H_n(M|A \cap B)$ satisfy the defining property of $\alpha_{A\cap B}$, hence must equal $\alpha_{A\cap B}$. Exactness of the sequence then implies that $(\alpha_A, -\alpha_B) = \Phi(\alpha_{A\cup B})$ for some $\alpha_{A\cup B} \in H_n(M|A \cup B)$. This means that $\alpha_{A\cup B}$ maps to α_A and α_B, so $\alpha_{A\cup B}$ has image α_x for all $x \in A \cup B$ since α_A and α_B have this property. To see that $\alpha_{A\cup B}$ is unique, observe that if a class $\alpha \in H_n(M|A \cup B)$ has image zero in $H_n(M|x)$ for all $x \in A \cup B$, then its images in $H_n(M|A)$ and $H_n(M|B)$ have the same property, hence are zero by hypothesis, so α itself must be zero since Φ is injective. Uniqueness of $\alpha_{A\cup B}$ follows by applying this observation to the difference between two choices for $\alpha_{A\cup B}$.

(2) Next we reduce to the case $M = \mathbb{R}^n$. A compact set $A \subset M$ can be written as the union of finitely many compact sets $A_1, \cdots, A_m$ each contained in an open $\mathbb{R}^n \subset M$. We apply the result in (1) to $A_1 \cup \cdots \cup A_{m-1}$ and A_m. The intersection of these two sets is $(A_1 \cap A_m) \cup \cdots \cup (A_{m-1} \cap A_m)$, a union of $m-1$ compact sets each contained in an open $\mathbb{R}^n \subset M$. By induction on m this gives a reduction to the case $m = 1$. When $m = 1$, excision allows us to replace M by the neighborhood $\mathbb{R}^n \subset M$.

(3) When $M = \mathbb{R}^n$ and A is a union of convex compact sets $A_1, \cdots, A_m$, an inductive argument as in (2) reduces to the case that A itself is convex, where the result is evident since the map $H_i(\mathbb{R}^n | A) \to H_i(\mathbb{R}^n | x)$ is an isomorphism for any $x \in A$, as both $\mathbb{R}^n - A$ and $\mathbb{R}^n - \{x\}$ deformation retract onto a sphere centered at x.

(4) For an arbitrary compact set $A \subset \mathbb{R}^n$ let $\alpha \in H_i(\mathbb{R}^n | A)$ be represented by a relative cycle z, and let $C \subset \mathbb{R}^n - A$ be the union of the images of the singular simplices in ∂z. Since C is compact, it has a positive distance δ from A. We can cover A by finitely many closed balls of radius less than δ centered at points of A. Let K be the union of these balls, so K is disjoint from C. The relative cycle z defines an element $\alpha_K \in H_i(\mathbb{R}^n | K)$ mapping to the given $\alpha \in H_i(\mathbb{R}^n | A)$. If $i > n$ then by (3) we have $H_i(\mathbb{R}^n | K) = 0$, so $\alpha_K = 0$, which implies $\alpha = 0$ and hence $H_i(\mathbb{R}^n | A) = 0$. If $i = n$ and α_x is zero in $H_n(\mathbb{R}^n | x)$ for all $x \in A$, then in fact this holds for all $x \in K$, where α_x in this case means the image of α_K. This is because K is a union of balls B meeting A and $H_n(\mathbb{R}^n | B) \to H_n(\mathbb{R}^n | x)$ is an isomorphism for all $x \in B$. Since $\alpha_x = 0$ for all $x \in K$, (3) then says that α_K is zero, hence also α, so the uniqueness statement in (a) is finished. The existence statement in (a) is easy since we can let α_A be the image of the element α_B associated to any ball $B \supset A$. ☐

The last paragraph in the proof of Proposition 3.29 should also be modified in the following way:

When $i = n$, the class $[z] \in H_n(M; R)$ defines a section $x \mapsto [z]_x$ of M_R. Since M is connected, this section is determined by its value at a single point, so $[z]_x$ must either be zero for all x or nonzero for all x. Since M is noncompact, $[z]_x$ must therefore be zero for all x since there are points x not in the compact image of z. By Lemma 3.27, z then represents zero in $H_n(M, V; R)$, hence also in $H_n(U; R)$ since the first term in the upper row of the diagram is zero when $i = n$, by Lemma 3.27 again. So $[z] = 0$ in $H_n(M; R)$ and hence $H_n(M; R) = 0$. ☐