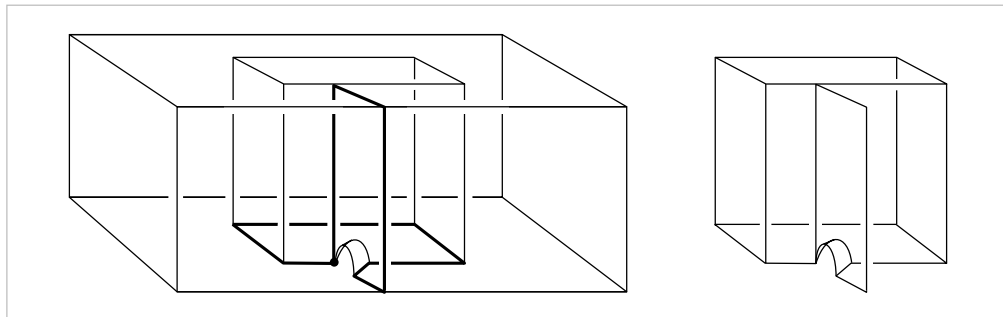


The House with One Room

The interesting feature of this 2-dimensional closed subspace of \mathbb{R}^3 is that it is contractible but not in any way that can easily be seen. To build this curious object,

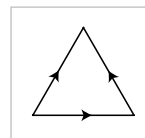


start with the six faces of a rectangular box. Remove a smaller open rectangle from the middle of the top face of the box and make its four edges the top edges of four vertical rectangles extending down to the bottom face of the box. What one has now is a sort of house with one room and an inner courtyard open to the sky. Next, add another vertical rectangle as a partition wall inside the room. Finally, a tunnel connecting the courtyard to the interior of the room is created by removing parts of two vertical rectangles and adding a curved triangle to form the walls and ceiling of the tunnel.

To see that this space X is contractible, consider a closed ε -neighborhood of it in \mathbb{R}^3 . This neighborhood N clearly deformation retracts onto X if ε is sufficiently small. In fact, N is the mapping cylinder of a map from the boundary surface of N to X . Less obvious is the fact that N is homeomorphic to D^3 , the closed unit ball in \mathbb{R}^3 . To see this, imagine forming N from a ball of clay by pushing a finger into the ball to create first the courtyard, then the tunnel, and then the inner room. Mathematically, this process gives a family of embeddings $h_t : D^3 \rightarrow \mathbb{R}^3$ starting with the usual inclusion $D^3 \hookrightarrow \mathbb{R}^3$ and ending with a homeomorphism onto N .

Thus X is a deformation retract of a ball. A ball also deformation retracts to a point, so X is homotopy equivalent to a point since homotopy equivalence is an equivalence relation. In fact, X itself deformation retracts to a point. For if f_t is a deformation retraction of the ball N to a point $x_0 \in X$ and if $r : N \rightarrow X$ is a retraction, for example the end result of a deformation retraction of N to X , then the restriction of the composition $r \circ f_t$ to X is a deformation retraction of X to x_0 . The reader who likes challenges might enjoy trying to see exactly what this deformation retraction looks like.

An easier but still interesting exercise is to show that X is homeomorphic to the quotient space of a triangle obtained by identifying all three of its edges together by homeomorphisms between them that preserve the orientations shown in the figure at the right. The classical



name for this space is the *dunce cap*. Identifying the left and right edges of the trian-

gle produces a cone, the traditional dunce cap shape, and then one imagines the cap sagging down on itself so that a line from its base to its tip is identified with the base circle. All three vertices of the triangle are identified to a single point, and the three edges become a circle. This circle is the heavy line in the earlier picture of the house with one room.