More Exercises for *Algebraic Topology* by Allen Hatcher

Chapter 0.

1. Given a map $f:X\to Y$, show that there exists a map $g:Y\to X$ with $gf \simeq 1\!\!1$ iff X is a retract of the mapping cylinder M_f.

2. (a) Suppose a CW complex X is the union of a finite number of subcomplexes X_i and that a subcomplex A of X is the union of subcomplexes $A_i \subset X_i$. Show that if each X_i deformation retracts onto A_i and each intersection of a subcollection of the X_i's deformation retracts onto the corresponding intersection of A_i's, then X deformation retracts onto A. [By an induction argument the problem reduces to the case of two X_i's and A_i's. In this special case show that the inclusions $A \hookrightarrow A \cup (X_1 \cap X_2) \hookrightarrow X$ are homotopy equivalences.]
(b) Use mapping cylinders to deduce the more general result that a map of CW complexes $f:X\to Y$ is a homotopy equivalence if it restricts to homotopy equivalences $X_{i_1} \cap \cdots \cap X_{i_k} \to Y_{i_1} \cap \cdots \cap Y_{i_k}$ for decompositions of X and Y as finite unions of subcomplexes $X_i \subset X$ and $Y_i \subset Y$ with $f(X_i) \subset Y_i$. Assume that f is a cellular map, sending n-skeleton to n-skeleton for all n. This guarantees that the mapping cylinder M_f is a CW complex. [The technique of cellular approximation described in §4.1 can be applied to show that the cellularity hypothesis can be dropped.]

3. Show that the n-skeleton of the simplex Δ^k has the homotopy type of a wedge sum of $\binom{k}{n+1}$ n-spheres.

4. For spaces $X \subset Y \subset Z$, show that X is a deformation retract of Y if Y is a retract of Z and Z deformation retracts onto X.

5. Suppose that a space X deformation retracts onto a subspace X_0 and we attach X to a space Y along a subspace $A \subset X_0$ via a map $f:A\to Y$ to form a space $Z = Y \sqcup_f X$. Show that Z deformation retracts onto $Z_0 = Y \sqcup_f X_0$.

Section 1.1.

1. If x_0 and x_1 are two points in the same path component of X, construct a bijection between the set of homotopy classes of paths from x_0 to x_1 and $\pi_1(X,x_0)$.

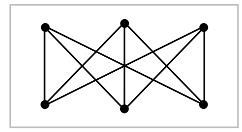
2. For spaces X and Y with basepoints x_0 and y_0 let $\langle X,Y\rangle$ denote the set of basepoint-preserving homotopy classes of basepoint-preserving maps $X\to Y$.
(a) Show that a homotopy equivalence $(Y,y_0) \simeq (Y',y_0')$ induces a bijection $\langle X,Y\rangle \approx \langle X,Y'\rangle$.
(b) Show that a homotopy equivalence $(X,x_0) \simeq (X',x_0')$ induces a bijection $\langle X,Y\rangle \approx \langle X',Y\rangle$.
(c) When X is a finite connected graph, compute $\langle X,Y\rangle$ in terms of $\pi_1(Y,y_0)$. [Use part (b) to reduce to the case that X is a wedge sum of circles.]

3. Show that if two maps $f, g : (X, x_0) \to (S^1, s_0)$ are homotopic just as maps $X \to S^1$ without regard to basepoints, then they are homotopic through basepoint-preserving maps via a homotopy $f_t : (X, x_0) \to (S^1, s_0)$. [Hint: Use rotations of S^1.]

Section 1.2.

1. Rederive the calculation $\pi_1(\mathbb{R}\mathrm{P}^2) \approx \mathbb{Z}_2$ using the CW structure on $\mathbb{R}\mathrm{P}^2$ obtained by identifying antipodal vertices, edges, and faces of a cube.

2. Let K be the graph with six vertices and nine edges shown at the right, and let X be obtained from K by attaching a 2-cell along each loop formed by a cycle of four edges in K. Show that $\pi_1(X) = 0$.



3. Let T be the torus $S^1 \times S^1$ and let T' be T with a small open disk removed. Let X be obtained from T by attaching two copies of T', identifying their boundary circles with longitude and meridian circles $S^1 \times \{x_0\}$ and $\{x_0\} \times S^1$ in T. Find a presentation for $\pi_1(X)$.

4. For X a finite connected graph, verify that $\pi_1(X \times S^1) \approx \pi_1(X) \times \pi_1(S^1)$ by computing $\pi_1(X \times S^1)$ using a CW structure on $X \times S^1$. In a similar fashion, verify that $\pi_1(X \times Y) \approx \pi_1(X) \times \pi_1(Y)$ when X and Y are wedge sums of circles.

5. Using a CW structure, compute the fundamental group of the mapping torus of the map $f : X \to X$ in the following cases:

(a) X is the graph formed by a circle with n equally-spaced radii and f is a rotation of this wheel graph sending each radius to the next.

(b) X is the graph which is the suspension of n points and f is the suspension of a cyclic permutation of the n points.

6. One way to modify a presentation $\langle g_1, \cdots, g_m \mid r_1, \cdots, r_n \rangle$ to another presentation for the same group is to replace a relation r_i by one of the products $r_i r_j$, $r_i r_j^{-1}$, $r_j r_i$, or $r_j^{-1} r_i$ for some $j \neq i$. Show that the 2-complexes X_G associated to these different presentations are homotopy equivalent. [Hint: Attach the 2-cell e_i^2 last and deform its attaching map so as to change r_i to one of the new relations, then apply Proposition 0.18.]

7. Apply the preceding problem to show that the complex X_G for the presentation $\langle a \mid a^p, a^q \rangle$ with $p, q > 1$ is homotopy equivalent to $S^2 \vee X_d$ where X_d is the complex associated to the presentation $\langle a \mid a^d \rangle$ for d the greatest common divisor of p and q.

8. Describe each of the following spaces as a mapping torus:

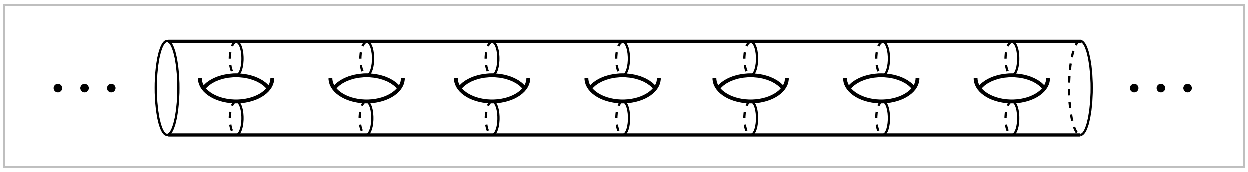
(a) $S^1 \times S^1$ with the identifications $(1, z) \sim (1, -z)$ and $(-1, z) \sim (-1, -z)$.

(b) $S^1 \times S^1$ with the identifications $(1, z) \sim (1, iz)$ and $(-1, z) \sim (-1, -z)$.

9. (a) Show that a finite CW complex, or more generally one with a finite 1-skeleton, has finitely generated fundamental group.

(b) Show that a map $f: X \to Y$ with X compact and Y a CW complex cannot induce an isomorphism on π_1 if $\pi_1(X)$ is not finitely generated. [Use part (a) and Proposition A.1 in the Appendix.]
(c) Deduce that the shrinking wedge of circles in Example 1.25 or an infinite product of circles is not homotopy equivalent to a CW complex.

10. [This is a revised version of Exercise 16 on page 54.] (a) Show that the noncompact surface of infinite genus shown below deformation retracts onto a graph, and use this to show that the fundamental group of the surface is free on a countably infinite number of generators.



(b) Do the same thing for the noncompact surface $\mathbb{R}^2 - C$ where C is the Cantor set in the x-axis.

11. Let X be the disk, annulus, or Möbius band, and let $\partial X \subset X$ be its boundary circle or circles.
(a) For $x \in X$ show that the inclusion $X - \{x\} \hookrightarrow X$ induces an isomorphism on π_1 iff $x \in \partial X$.
(b) If Y is also a disk, annulus, or Möbius band, show that a homeomorphism $f: X \to Y$ restricts to a homeomorphism $\partial X \to \partial Y$.
(c) Deduce that the Möbius band is not homeomorphic to an annulus.

Section 1.3.

1. Construct an uncountable number of nonisomorphic connected covering spaces of $S^1 \vee S^1$. Deduce that a free group on two generators has an uncountable number of distinct subgroups. Is this also true of a free abelian group on two generators?

2. [An expanded version of number 4 on page 79.] Construct a simply-connected covering space for each of the following spaces: (a) $S^1 \vee S^2$. (b) The union of S^2 with an arc joining two distinct points of S^2. (c) S^2 with two points identified. (d) $\mathbb{R}P^2 \vee \mathbb{R}P^2$. (e) S^2 with two arcs joining two pairs of points, or the same pair of points. (f) $S^1 \vee \mathbb{R}P^2$. (g) $\mathbb{R}P^2$ with an arc joining two distinct points. (h) $S^1 \vee T^2$ where T^2 is the torus.

3. Describe geometrically the structure of an arbitrary covering space of $S^1 \vee \mathbb{R}P^2$.

4. If N_g denotes the closed nonorientable surface defined in §1.2, construct an n-sheeted covering space $N_{mn+2} \to N_{m+2}$ for each $m \geq 0$ and $n \geq 2$.

5. [An addition to number 18 on page 80.] Determine explicitly all the abelian covering spaces of $S^1 \vee S^1$. How are these related to covering spaces of $S^1 \times S^1$?

6. (a) Show that a map $f: X \to Y$ between Hausdorff spaces is a covering space if X is compact and f is a local homeomorphism, meaning that for each $x \in X$ there are

open neighborhoods U of x in X and V of $f(x)$ in Y with f a homeomorphism from U onto V.
(b) Give an example where this fails if X is noncompact.

7. [A second part to number 6 on page 79.] Show that the composition of two covering spaces is a covering space if the second one is finite-sheeted.

8. Show that the noncompact surface of infinite genus shown in Exercise 16 for §1.2 is a covering space of the closed orientable surface of genus 2 in two different ways, one way via an action of $\mathbb{Z}$ on the noncompact surface and the other way via an action of the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$.

9. Using covering spaces, show that a finite index subgroup of a finitely generated group is finitely generated, and similarly with 'finitely generated' replaced by 'finitely presented' (finite number of generators and relations). Is there a bound on the number of generators of the subgroup in terms of the index and the number of generators of the full group?

10. A third part of Exercise 21 on page 81: Do the same for the space obtained from Y by attaching a second Möbius band along the same circle that the first one was attached along.

11. Another part to Exercise 16 on page 80, showing the necessity of the hypothesis that Z be locally path-connected in the first part of the exercise: Let Z be the subspace of $\mathbb{R}$ consisting of the sequence $x_n = 1/n$, $n = 1, 2, \cdots$, together with its limit 0. Let $X \subset Z \times \{0, 1, 2, \cdots\} \subset \mathbb{R}^2$ consist of the pairs $(0, k)$, the pairs $(x_n, 2j+1)$ with $n \geq 1$ and $j \geq 0$, and the pairs $(x_n, 2j)$ with $n > j \geq 0$. Let $Y = Z \times \{0, 1\} \subset X$. Reducing the second coordinate modulo 2 gives a map $X \to Y$, and then forgetting the second coordinate gives a map $Y \to Z$. Show that the composition $X \to Y \to Z$ is a covering space, as is the map $Y \to Z$, but the map $X \to Y$ is not a covering space.

12. Give an example of a map $p : \widetilde{X} \to X$ that is not a covering space but satisfies the covering homotopy property, including uniqueness of the lift.

13. (Exercise on non-pathconnected covering spaces) Assume X is locally pathconnected. (a) Show that a map $p : \widetilde{X} \to X$ is a covering space if and only if the restriction $p : p^{-1}(X_0) \to X_0$ is a covering space for each component X_0 of X, and do the same for normal covering spaces. (b) Now assume X is connected as well as locally pathconnected. Show that if $p : \widetilde{X} \to X$ is a covering space then so is the restriction $p : \widetilde{X}_0 \to X$ for each path component $\widetilde{X}_0$ of $\widetilde{X}$, and give a counterexample where the converse fails. Do the same also for normal covering spaces.

14. Let X be one of the five graphs formed by the vertices and edges of a regular polyhedron: tetrahedron, cube, octahedron, dodecahedron, or icosahedron. Determine the values of $n > 1$ such that there exists an n-sheeted covering space $X \to Y$. [Hint: A finite graph must have an odd number of odd-valence vertices, where the valence of a vertex means the number of edges that abut it, using the convention that if both ends of an edge abut the same vertex, this contributes 2 to the valence.]

15. Let X be a path-connected Hausdorff space. Show that a covering space $\widetilde{X} \to X$ must be finite-sheeted if $\widetilde{X}$ is compact. Deduce that $\pi_1(X)$ is finite if X has a compact simply-connected covering space.

16. [An enhanced version of Exercise 20 on page 81.] Let K be the Klein bottle and T the torus.
(a) Construct an n-sheeted normal covering space $K \to K$ for each $n > 1$.
(b) Construct an n-sheeted nonnormal covering space $K \to K$ for each $n > 2$. [Note that 2-sheeted covering spaces are normal since index-two subgroups are normal.]
(c) Show that a covering space $T \to K$ must have an even number of sheets.
(d) Construct an n-sheeted normal covering space $T \to K$ for each even $n > 1$.
(e) Construct an n-sheeted nonnormal covering space $T \to K$ for each even $n > 2$.

17. For p prime, find all the index p normal subgroups of $\mathbb{Z} * \mathbb{Z}$ and the corresponding covering spaces of $S^1 \vee S^1$.

18. [An addendum to problem 14 on page 80.] Classify the subgroups of $\mathbb{Z}_2 * \mathbb{Z}_2$ up to isomorphism, showing there are only four possibilities. How many isomorphism types of subgroups of $\mathbb{Z}_2 * \mathbb{Z}_3$ and $\mathbb{Z}_3 * \mathbb{Z}_3$ are there?

Section 1.B.

1. Addition to Exercise 5: How does the universal cover change if the relator $bab^{-1}a^{-2}$ is replaced by $bab^{-1}a^2$?

Section 2.1.

1. Compute the simplicial homology groups of S^1 with the Δ-complex structure having n vertices and n edges.

2. Show that the simplicial homology groups of an oriented graph do not depend on the orientations of the edges.

3. Regarding Δ^n as a Δ-complex in the natural way, show that if a subcomplex $X \subset \Delta^n$ has $H_{n-1}(X)$ nonzero then $X = \partial\Delta^n$.

4. A map $f: X \to Y$ induces a function from the set of path-components of X to the set of path-components of Y. Show that this function determines and is determined by the induced homomorphism $f_*: H_0(X) \to H_0(Y)$.

5. Compute the local homology groups of the mapping cylinders of the maps $\mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin x$ and $g(x) = x \sin 1/x$.

6. Show that the local homology groups of a finite simplicial complex are finitely generated, and construct a finite CW complex having a local homology group that is not finitely generated. [See the previous problem.]

7. Show that the obvious quotient map from the augmented chain complex of a space X to the unaugmented chain complex is a chain map. Thus this map is part of a short exact sequence of chain complexes, with kernel the complex having only a $\mathbb{Z}$ in dimension -1, so there is an induced long exact sequence of homology groups which includes the short exact sequence $\widetilde{H}_0(X) \to H_0(X) \to \mathbb{Z} \to 0$. Observe that these various sequences are all natural.

8. There are exactly three ways to identify the faces of Δ^3 in pairs to produce a Δ-complex. Compute the homology groups of this Δ-complex in each case. [This is related to exercise 7 on page 131.]

9. (a) Show that the set $\{\,(x_1, \cdots, x_n) \in \mathbb{R}^n \mid 0 \le x_1 \le \cdots \le x_n \le 1\,\}$ is a simplex of dimension n.
(b) Show that the n-dimensional cube I^n has the structure of a simplicial complex with the same set of 2^n vertices as the cube and with $n!$ n-simplices.

10. Given a set S, let X_S be the simplicial complex whose vertices are the elements of S and whose simplices are all the finite subsets of S. Show that X_S is contractible if $S \ne \varnothing$.

Section 2.2.

1. Here is a corrected and extended version of problem 13 on page 156. Let X be the 2-complex obtained from S^1 with its usual cell structure by attaching two 2-cells by maps of degrees 2 and 3, respectively.

(a) Compute the homology groups of all the subcomplexes $A \subset X$ and the corresponding quotient complexes X/A.

(b) Show that the only subcomplex $A \subset X$ such that the quotient map $X \to X/A$ is a homotopy equivalence is the trivial subcomplex consisting of the 0-cell alone.

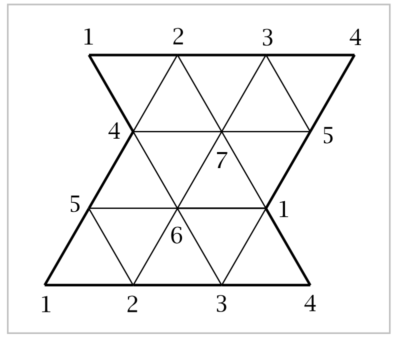
(c) Show that $X \simeq S^2$. [Hint: Use Proposition 0.18 and consider the possibility of attaching one 2-cell after the other and then deforming its attaching map.]

(d) Let Y be obtained from X by attaching a 3-cell by a map $S^2 \to X$ that is the composition of a degree 2 map of S^2 with a homotopy equivalence $S^2 \to X$. Show that if B is a nontrivial subcomplex of Y then Y and Y/B are not homotopy equivalent.

2. Show that if X is a CW complex with k n-cells then $H_n(X) \approx \mathbb{Z}^k$ iff the cellular boundary maps d_n and d_{n+1} are both zero.

3. Use Euler characteristic to determine which orientable surface results from identifying opposite edges of a $2n$-gon.

4. Suppose a simplicial complex structure on a closed surface of Euler characteristic χ has v vertices, e edges, and f faces, which are triangles. Show that $e = 3f/2$, $f = 2(v - \chi)$, $e = 3(v - \chi)$, and $e \le v(v-1)/2$. Deduce that $6(v - \chi) \le v^2 - v$. For the torus conclude that $v \ge 7$, $f \ge 14$, and $e \ge 21$. Explain how the diagram at the right gives a simplicial complex structure on the torus realizing the minimum values $(v, e, f) = (7, 21, 14)$. For the projective plane show that $v \ge 6$, $f \ge 10$, and $e \ge 15$, and use the icosahedron to describe a simplicial complex structure realizing the minimum values $(v, e, f) = (6, 15, 10)$. Why does the octahedron not work?



5. The degree of a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ can be defined as the degree of the extension of f to a homeomorphism of the one-point compactification S^n. Using this notion, fill in the details of the following argument due to R. Fokkink which shows that $\mathbb{R}^n$ is not homeomorphic to a product $X \times X$ if n is odd. Assuming $\mathbb{R}^n = X \times X$, consider the homeomorphism f of $\mathbb{R}^n \times \mathbb{R}^n = X \times X \times X \times X$ that cyclically permutes the factors, $f(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4, x_1)$. Then f^2 switches the two factors of $\mathbb{R}^n \times \mathbb{R}^n$, so f^2 has degree -1 if n is odd. But $\deg(f^2) = (\deg f)^2 = +1$.

6. Show that the homology groups of a join $X * Y$ are given by $\widetilde{H}_n(X * Y) \approx H_{n-1}(X \times Y, X \vee Y)$.

7. For a simplicial complex X construct a Δ-complex $\Delta(X)$ having an n-simplex for each linear map of Δ^n to a simplex of X that takes vertices to vertices but not necessarily injectively. The faces of such a simplex of $\Delta(X)$ are obtained by restricting the map to faces of Δ^n. Show that the natural map $\Delta(X) \to X$ induces an isomorphism on all homology groups. Thus the homology of X can be computed using the simplicial chain complex of $\Delta(X)$, which lies between the simplicial and singular chain complexes of X. [First do the case that X is a simplex, then use a Mayer-Vietoris argument to do the case that X is a finite complex by induction on the number of simplices, then deduce the general case.]

8. Suppose the definition of $\Delta(X)$ in the preceding exercise was modified to allow only injective linear maps $\Delta^n \to X$. Show that in the cases $X = \Delta^1$ and $X = \Delta^2$ it is no longer true with the more restrictive definition that X and $\Delta(X)$ have isomorphic homology groups.

9. Let S^n be given the CW structure lifting the standard CW structure on $\mathbb{R}\mathrm{P}^n$, so that S^n has two i-cells for each $i \le n$. Compute the resulting cellular chain complex for S^n and verify that it has the correct homology groups. [Use orientations for the cells that lift orientations of the cells of $\mathbb{R}\mathrm{P}^n$.]

10. Compute the homology groups of the quotient space of S^n obtained by identifying antipodal points in the standard $S^k \subset S^n$, for a fixed $k < n$.

11. Compute the reduced Mayer-Vietoris sequence for the CW complex $X = A \cup B$ where $A \cap B$ is $\mathbb{R}P^2$ and we obtain A from $A \cap B$ by attaching a cone on $\mathbb{R}P^1 \subset \mathbb{R}P^2$ and we obtain B by attaching a cone on all of $\mathbb{R}P^2$. Thus $A \simeq S^2$, B is contractible, and $X \simeq S^2$. [The Mayer-Vietoris sequence reduces to a short exact sequence that does not split. This gives a counterexample to the original formulation of the Mayer-Vietoris sequence by Mayer and Vietoris, which was not in terms of exact sequences since these had not yet been invented, but rather as a direct sum statement. The error is repeated more recently in [Dieudonné 1989], page 39.]

12. (a) Compute the homology groups of the quotient space of the unit sphere $S^2 \subset \mathbb{R}^3$ obtained by identifying $(x,y,z) \sim (y,z,x) \sim (z,x,y)$ whenever at least one of x,y,z is 0. [To get a cell structure, replace S^2 by a regular octahedron centered at the origin with vertices on the coordinate axes.]
(b) Determine whether this space is homotopy equivalent to the wedge sum of Moore spaces giving the same homology.

13. Show that if (X,A) is a CW pair of dimension n (so all cells of $X - A$ have dimension at most n) then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \hookrightarrow X$ is injective with image a direct summand of $H_n(X)$.

14. Does the singular chain defined by the quotient map $\Delta^n \to \Delta^n/\partial\Delta^n = S^n$ represent a generator of $H_n(S^n)$ for $n > 0$? If not, what can be added to get a generator?

15. Show that if $f : \Delta^n \to \Delta^n$ is a map that takes each $(n-1)$-dimensional face of Δ^n to itself, then f is surjective. [Consider the induced map $\Delta^n/\partial\Delta^n \to \Delta^n/\partial\Delta^n$.]

16. Show that the spaces $S^1 \times S^2$ and $S^1 \vee S^2 \vee S^3$ have isomorphic homology and fundamental groups but are not homotopy equivalent. [Compute the homology groups of their universal covers.]

17. Give a proof or a counterexample to the following statement, for each $n > 0$: If X is an n-dimensional CW complex having exactly one cell in each dimension $0, 1, \cdots, n$ and $H_i(X) \approx \mathbb{Z}$ for $0 \le i \le n$, then $X \simeq S^1 \vee S^2 \vee \cdots \vee S^n$.

Section 2.B.

1. Generalizing Corollary 2B.4, show that if a map from a compact manifold to a connected manifold of the same dimension is locally an embedding, then it is a covering space.

2. Using Proposition 2B.1, show that a subspace $X \subset S^n$ homeomorphic to S^{n-1} is the frontier of each of the two components of $S^n - X$. [Observe that each point $x \in X$ has arbitrarily small open neighborhoods N in X such that $S^n - (X - N)$ is connected, then consider a path in $S^n - (X - N)$ connecting a point in one component of $S^n - X$ to a point in the other component.]

Section 3.1.

1. Write down an explicit cocycle in $C^1(S^1;\mathbb{Z})$ representing a generator of $H^1(S^1;\mathbb{Z})$.

Section 3.2.

1. Compute the ring $H^*(X;\mathbb{Z})$ when X is obtained from $\mathbb{C}\mathrm{P}^{n-1}$ by attaching a cell e^{2n} by the composition $S^{2n-1} \xrightarrow{f} S^{2n-1} \xrightarrow{g} \mathbb{C}\mathrm{P}^{n-1}$ where f has degree $k > 1$ and g is the attaching map for the $2n$-cell of $\mathbb{C}\mathrm{P}^n$.

2. Compute the ring $H^*(X;\mathbb{Z})$ when X is obtained from $\mathbb{C}\mathrm{P}^{n-1} \vee S^{2n-1}$ by attaching a cell e^{2n} by the composition $S^{2n-1} \to S^{2n-1} \vee S^{2n-1} \xrightarrow{f \vee g} S^{2n-1} \vee \mathbb{C}\mathrm{P}^{n-1}$ where the first map collapses the equator $S^{2n-2} \subset S^{2n-1}$ to a point, the map f has degree $k > 1$, and g is the attaching map for the $2n$-cell of $\mathbb{C}\mathrm{P}^n$.

3. Generalizing Example 3.7, compute the ring $H^*(X;\mathbb{Z})$ where X is obtained from a wedge sum of finitely many circles by attaching a 2-cell according to an arbitrary product of commutators of the 1-cells. More generally, do this for any attaching map for which $H^2(X;\mathbb{Z}) \approx \mathbb{Z}$.

4. Show that if there exists a map $f: S^n \times S^n \to S^n$ that is an odd function of each variable separately, so $f(-x,y) = -f(x,y) = f(x,-y)$, then $n = 2^k - 1$ for some integer k. [See Theorem 3.20.]

5. For a space X let $c(X)$ denote the minimum number of sets in a covering of X by contractible open sets, if such a cover exists. Using Exercise 2 on page 228 show:
(a) $c(\mathbb{R}\mathrm{P}^n) = c(\mathbb{C}\mathrm{P}^n) = n + 1$.
(b) $c(M) = 3$ for closed surfaces M other than S^2.
(c) $c(T^n) = n + 1$ for T^n the n-dimensional torus.

6. From the calculation $H^*((\mathbb{R}\mathrm{P}^\infty)^n;\mathbb{Z}_2) \approx \mathbb{Z}_2[x_1, \cdots, x_n]$ deduce that all cellular boundary and coboundary maps with $\mathbb{Z}_2$ coefficients are zero for the product $(\mathbb{R}\mathrm{P}^\infty)^n$ with its standard CW structure.

7. Taking the product of the inclusion $\mathbb{R}\mathrm{P}^1 \hookrightarrow \mathbb{R}\mathrm{P}^\infty$ with itself n times gives a map from the n-torus T^n to $(\mathbb{R}\mathrm{P}^\infty)^n$. Compute the induced maps on $H^*(-;\mathbb{Z}_2)$ and $H_*(-;\mathbb{Z}_2)$.

8. By Proposition 1B.9 there is a map $\mu: \mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty \to \mathbb{R}\mathrm{P}^\infty$ whose induced map on π_1 is the multiplication map in this group. Compute the maps induced by μ on $H^*(-;\mathbb{Z}_2)$ and $H_*(-;\mathbb{Z}_2)$, showing that the latter map takes the cellular homology class of the product cell $e^i \times e^j$ to $\binom{i+j}{i} e^{i+j}$ where the binomial coefficient is taken $\bmod 2$. [This can be interpreted as saying that $H_*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$ is a divided polynomial algebra using the multiplication induced by μ. See §3.C for more on this idea.]

9. Deduce the cup product structure for a product of spheres from the special case of the n-torus T^n using the fact that S^i is T^i with its $(i-1)$-skeleton collapsed to a point, where T^i is given its standard CW structure.

Section 3.3.

1. Show that the degree of a map $f: T^n \to T^n$ of the n-dimensional torus is the determinant of $f^*: H^1(T^n;\mathbb{Z}) \to H^1(T^n;\mathbb{Z})$.

2. For a connected nonorientable manifold show that a two-sheeted covering space that is orientable is unique up to isomorphism.

3. Let $p:\widetilde{M}\to M$ be the two-sheeted orientable cover of the nonorientable closed n-manifold M. Show that $H_k(\widetilde{M};F) \approx H_k(M;F)\oplus H_{n-k}(M;F)$ for $F=\mathbb{Q}$ or $\mathbb{Z}_p$ with p an odd prime, by filling in details in the following outline:

(a) For a vector space V over F let $T:V\to V$ be a linear map with $T^2=\mathbb{1}$, and let $V^{\pm}=\{v\in V \mid T(v)=\pm v\}$. Show that $V=V^+\oplus V^-$ using the formula $v={}^1\!/_2[v+T(v)]+{}^1\!/_2[v-T(v)]$.

(b) In particular, there are splittings $H_k(\widetilde{M};F)=H_k^+(\widetilde{M};F)\oplus H_k^-(\widetilde{M};F)$ induced by the nontrivial deck transformation $\tau:\widetilde{M}\to\widetilde{M}$, and similarly for cohomology, with the latter splitting being the Hom-dual of the former.

(c) Identify $H_k(M;F)$ with $H_k^+(\widetilde{M};F)$ and likewise for cohomology, by associating to each singular simplex in M the sum of its two lifts to $\widetilde{M}$.

(d) Show that the Poincaré duality isomorphism $\alpha\mapsto[\widetilde{M}]\frown\alpha$ identifies the $+$ and $-$ parts of $H^k(\widetilde{M};F)$ with the $-$ and $+$ parts of $H_{n-k}(\widetilde{M};F)$, respectively, using the fact that $\tau_*[\widetilde{M}]=-[\widetilde{M}]$.

4. Using Poincaré duality and the naturality property of cap products, show that a map $f:M\to N$ of degree 1 between closed orientable n-manifolds induces split surjections $f_*:H_i(M;R)\to H_i(N;R)$ for all i.

5. Let M be a closed 3-manifold embedded in S^4 so that it has a neighborhood homeomorphic to $M\times(-\varepsilon,\varepsilon)$, with M corresponding to $M\times\{0\}$ under this homeomorphism. Show that the torsion subgroup of $H_1(M)$ splits as a direct sum $T\oplus T$ for some finite abelian group T.

6. Show that the space $X=\mathbb{R}^2-\{(x,0)\mid x\neq 0\}$ is not a manifold, nor is the product $X\times\mathbb{R}^k$ for any k.

Section 3.B.

1. Show that S^n is not homeomorphic to a product $X\times Y$ unless X or Y is a point. [Hint: If $S^n=X\times Y$ then X and Y embed in S^n as retracts.]

Section 3.C.

1. Apply Proposition 1B.9 to show that if G is an abelian group then a CW complex $K(G,1)$ is a homotopy-associative, homotopy-commutative H–space.

2. An additional part (d) for the existing problem 10: In case $H^*(X;R)$ is an exterior algebra $\Lambda_R[\alpha_1,\cdots,\alpha_r]$ on r odd-dimensional generators α_i show that the k^{th}-power map $x\mapsto x^k$ induces multiplication by k^r on the top-dimensional cohomology group $H^n(X;R)$ for $n=|\alpha_1|+\cdots+|\alpha_r|$.

Section 4.1.

1. For X path-connected show that $\pi_n(X)=0$ iff every pair of maps $f_0,f_1:D^n\to X$

with $f_0|\partial D^n = f_1|\partial D^n$ is joined by a homotopy $f_t: D^n \to X$ with $f_t|\partial D^n = f_0|\partial D^n$ for all t.

2. Let $X \subset \mathbb{R}$ consist of the sequence $1/2, 1/3, 1/4, \cdots$ together with its limit 0. Show that the suspension SX has $\pi_n(SX) = 0$ for all $n > 1$.

3. Let $f: S^n \times S^n \to S^{2n}$ be the quotient map collapsing $S^n \vee S^n$ to a point. Show that f induces the zero map on all homotopy groups but f is not nullhomotopic.

4. Let (X, A) be an n-connected CW pair with A of dimension less than n. Show that if (X', A') is another such pair with $X \simeq X'$ then $A \simeq A'$. Give an example where this fails when A has dimension n.

5. Show that for a pair (X, A), the image of the map $\pi_2(X, x_0) \to \pi_2(X, A, x_0)$ lies in the center of $\pi_2(X, A, x_0)$. (This was Exercise 27 in Section 4.2, but it really belongs in Section 4.1 since it can be proved directly from the definitions.)

6. For a path-connected space X with suspension SX show that there are isomorphisms $\pi_n(SX, X) \approx \pi_n(SX) \times \pi_{n-1}(X)$ for all $n \geq 2$. [The case $n = 2$ needs special attention since it can involve nonabelian groups.]

7. As a second part to exercise 13 on page 359, use the Brouwer fixed point theorem to show that every map from a contractible finite simplicial complex to itself has a fixed point.

8. For path-connected X and Y, use the long exact sequence of homotopy groups for the pair $(X \times Y, X \vee Y)$ to show that $\pi_n(X \vee Y) \approx \pi_n(X) \times \pi_n(Y) \times \pi_{n+1}(X \times Y, X \vee Y)$ for all $n \geq 2$.

Section 4.2.

1. A space Y is a *homotopy retract* of a space X if there are maps $Y \xrightarrow{i} X \xrightarrow{r} Y$ whose composition is homotopic to the identity. Show that if a simply-connected CW complex Y is a homotopy retract of a wedge of spheres $\bigvee_i S^{n_i}$, for example if $\bigvee_i S^{n_i} \simeq Y \vee Z$ for some Z, then Y is also homotopy equivalent to a wedge of spheres. [Hint: If the Hurewicz map $\pi_n \to H_n$ is surjective for a space X then it is also surjective for any homotopy retract of X. Use this to construct a map from a wedge of spheres to Y inducing an isomorphism on homology.]

2. Show that if X is m-connected and Y is n-connected then the join $X * Y$ is $(m+n+2)$-connected. This also holds when m or n is -1 if (-1)-connected is taken to mean nonempty. [Simple-connectivity is an exercise for §1.2. For higher connectivity use the Hurewicz theorem and the fact that $\widetilde{H}_i(X * Y)$ is isomorphic to $H_{i-1}(X \times Y, X \vee Y)$, an exercise for §2.2.]

3. Give examples of maps between simply-connected CW complexes that induce a surjection on π_* but not on H_*, and vice versa, and do the same for injections. Four examples are needed in total. The spaces can be chosen to be spheres and complex projective spaces.

4. Let $F \longrightarrow E \xrightarrow{p} B$ be a fiber bundle with base B a finite CW complex. Show that if $H_*(F;\mathbb{Z})$ is finitely generated then so is $H_*(E;\mathbb{Z})$ and $\chi(E) = \chi(F)\chi(B)$. [Proceed by induction on the number of cells in B. Write $B = C \cup D^n$ where D^n is a disk in a top-dimensional cell e^n of B such that the bundle is a product over D^n and $C \cap D^n = \partial D^n$. Show the inclusion $p^{-1}(B - e^n) \hookrightarrow p^{-1}(C)$ is a weak homotopy equivalence, and use the Mayer-Vietoris sequence for the decomposition $E = p^{-1}(C) \cup p^{-1}(D^n)$.]

5. Let $C_f = S^n \cup e^k$ where the cell e^k is attached by $f: S^{k-1} \to S^n$, and let C_g be constructed similarly using $g: S^{k-1} \to S^n$. Show that if $C_f \simeq C_g$ then g is homotopic to the composition of f with homotopy equivalences of S^{k-1} and S^n. In particular, if $C_f \simeq S^n \vee S^k$ then f is nullhomotopic.

6. Let $f:(X,A) \to (Y,B)$ be a map of CW pairs such that both the restriction $f:A \to B$ and the induced quotient map $X/A \to Y/B$ are homotopy equivalences. Show that $f: X \to Y$ is a homotopy equivalence if X and Y are simply-connected. Show also, by means of an example of a map $(S^1 \vee S^n, S^1) \to (S^1 \vee S^n, S^1)$ with $n > 1$, that the simple-connectivity assumption cannot be dropped.

7. Show that if a map $f: X \to Y$ of CW complexes induces isomorphisms on homology $f_*: H_n(X) \to H_n(Y)$ for all n, then the suspension $Sf: SX \to SY$ is a homotopy equivalence, even when X and Y are not connected.

8. Show that if G is a finite group having a finite presentation with the same number of generators as relations, then $H_2(G) = 0$, where $H_i(G)$ means $H_i(K(G,1))$ for any $K(G,1)$ CW complex.

9. Show that for X path-connected the suspension map $\pi_1(X) \to \pi_2(SX)$ is abelianization.

Section 4.3.

1. Show that associating to a map $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ the induced homomorphism on H_2 gives a bijection $[\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n] \approx \mathbb{Z}$.

2. A short exact sequence of groups $1 \to A \hookrightarrow B \to C \to 1$ corresponds to a fibration $K(A,1) \to K(B,1) \to K(C,1)$. Show this fibration is principal if and only if A lies in the center of B.

3. Construct a pair (X,Y) such that $\pi_1(Y)$ is abelian but $\pi_2(X,Y)$ is nonabelian, by applying the preceding problem in a case where A is in the center of the nonabelian group B with abelian quotient C (for example, B is the quaternion group of order 8 and A is its center), using the general fact that $\pi_{n+1}(X,Y)$ is π_n of the homotopy fiber of the inclusion $Y \hookrightarrow X$.

4. (a) Let $p: E \to B$ be a fibration and let $r: E \to E_0$ be a retraction onto a subspace $E_0 \subset E$ such that $pr = p$ (so r preserves fibers of p). Show that the restriction $p: E_0 \to B$ is also a fibration.

(b) Use part (a) to give simple examples of fibrations which are not fiber bundles, with E a product $B \times F$ and, say, $B = F = I$. [This gives a simpler solution to Exercise 9 on page 419 not using the hint given there.]

Section 4.D.

1. Prove the following refinement of the Leray-Hirsch theorem: Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fiber bundle such that, for some commutative coefficient ring R and some fixed integer $m > 0$ the R-modules $H^j(F;R)$ for $j \le m$ are free with a finite basis consisting of the restrictions $i^*(c_{jk})$ of elements $c_{jk} \in H^j(E;R)$. Then the map $\Phi : \bigoplus_j (H^{\ell-j}(B;R) \otimes_R H^j(F;R)) \to H^\ell(E;R)$, $\sum_{ijk} b_i \otimes i^*(c_{jk}) \mapsto \sum_{ijk} p^*(b_i) \smile c_{jk}$, is an isomorphism for $\ell \le m$.

Section 4.K.

1. Show that a map $p : E \to B$ is a quasifibration if it satisfies a weaker form of the homotopy lifting property which allows a homotopy $D^k \times I \to B$ to be reparametrized by composition with a map $D^k \times I \to D^k \times I$ of the form $(x,t) \mapsto (x, g_x(t))$ for some family of maps $g_x : (I,0,1) \to (I,0,1)$ before lifting the homotopy.

Section 4.L.

1. Using the argument in Proposition 4L.11 together with the Adem relations $Sq^1Sq^4 + Sq^2Sq^3 + Sq^4Sq^1 = 0$, $Sq^1Sq^8 + Sq^2Sq^7 + Sq^8Sq^1 = 0$, and $Sq^2Sq^8 + Sq^4Sq^6 + Sq^8Sq^2 + Sq^9Sq^1 = 0$, show that 2ν, 2σ, and $\sigma\eta$ are nonzero in π_*^s. Explain why the argument fails for the products $\nu\eta$ and $\sigma\nu$, where the Adem relations in question are $Sq^2Sq^4 + Sq^5Sq^1 + Sq^6 = 0$ and $Sq^4Sq^8 + Sq^{10}Sq^2 + Sq^{11}Sq^1 + Sq^{12} = 0$. However the argument does work unstably for the compositions $\nu\eta : S^8 \to S^7 \to S^4$ and $\sigma\nu : S^{18} \to S^{15} \to S^8$, together with a few suspensions of these compositions. [In fact $\nu\eta$ and $\sigma\nu$ are both zero in π_*^s.]

Appendix

1. A CW complex is said to be *countable* if it has countably many cells, and it is *locally finite* if every point has a neighborhood that is contained in a finite subcomplex. The latter condition is equivalent to being locally compact.

(a) Show that every countable CW complex can be expressed as the union of an increasing sequence of finite subcomplexes $X_1 \subset X_2 \subset \cdots$.

(b) Show that every countable CW complex is homotopy equivalent to a locally finite CW complex. [Consider the mapping telescope of the inclusions in (a); see the proof of Lemma 2.34.] Refine the construction to show that when the given complex is finite-dimensional, the locally finite complex can be taken to be of the same dimension.

(c) Show that a locally finite CW complex that is connected must be countable.