Math 651 — Problem Set 2 — Spring 2005

Terminology used in some of the problems: "nullhomotopic" means "homotopic to a constant map".

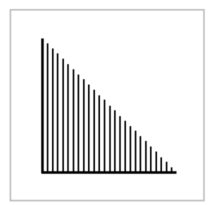
1. Show that if $\varphi : X \to Y$ is a homotopy equivalence (without any conditions on basepoints) then the induced homomorphisms $\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, \varphi(x_0))$ are isomorphisms for all n. [Generalize the proof of Proposition 1.18 in the book, which is the case $n = 1$.]

2. Construct a deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus. [Hint: Regard the torus as a quotient space of a square with opposite sides identified.]

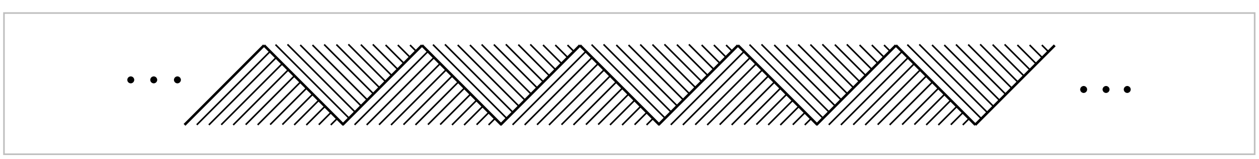
3. A *deformation retraction in the weak sense* of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 = \mathbb{1}$, $f_1(X) \subset A$, and $f_t(A) \subset A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

4. Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of x in X there exists a neighborhood $V \subset U$ of x such that the inclusion map $V \hookrightarrow U$ is nullhomotopic.

5. (a) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point. [See the preceding problem.]



(b) Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the figure below. Show that Y is contractible but does not deformation retract onto any point.



(c) Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$ indicated by the heavier line. Show there is a deformation retraction in the weak sense (see Exercise 3) of Y onto Z, but no true deformation retraction. [Hint: If a space A deformation retracts onto a subspace B and B deformation retracts onto a subspace C, then A deformation retracts onto C.]

6. Show that a space X is homotopy equivalent to a point if and only if the identity map $X \to X$ is nullhomotopic. [These are the two equivalent conditions for a space to be contractible.]

7. Show that a retract of a contractible space is contractible.

8. Show that a space X is contractible iff every map $f : X \to Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible iff every map $f : Y \to X$ is nullhomotopic.

9. Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $fg \simeq \mathbb{1}$ and $hf \simeq \mathbb{1}$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalences.

10. Let E be a subspace of $\mathbb{R}^2$ obtained by deleting a subspace of $\{0\} \times \mathbb{R}$. For which such spaces E is the projection $E \to \mathbb{R}$, $(x, y) \mapsto x$, a fiber bundle?

11. For a map $f : X \to X$ the quotient space T_f of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$ is called the *mapping torus* of f. Show that if f is a homeomorphism, then the map $T_f \to S^1$ induced from the projection $(x, t) \mapsto t$ is a fiber bundle with fiber X.

12. For a pair (X, A) of path-connected spaces, show that $\pi_1(X, A, x_0)$ can be identified in a natural way with the set of cosets αH of the subgroup $H \subset \pi_1(X, x_0)$ represented by loops in A at x_0.