1. Let $K$ be the graph with six vertices and nine edges shown at the right, and let $X$ be obtained from $K$ by attaching a 2-cell along each loop formed by a cycle of four edges in $K$.

(a) Show that $\pi_1(X) = 0$.

(b) Compute the Euler characteristic $\chi(X)$.

(c) Compute the groups $H_n(X)$, without using the general fact that $H_1(X)$ is the abelianization of $\pi_1(X)$. Hint: Using (b), this can be done with very little actual calculation.

(d) Do parts (a), (b), (c) when $X$ is the 2-skeleton of the simplex $\Delta^4$, and also when $X$ is the 3-skeleton of $\Delta^4$.

2. Use Euler characteristic to determine which orientable surface results from identifying opposite edges of a polygon with $2n$ sides. (You can use the fact that closed orientable surfaces are determined up to homeomorphism by their Euler characteristic.)

3. Let $X$ be the 2-complex obtained from $S^1$ with its usual cell structure (one 0-cell and one 1-cell) by attaching two 2-cells by maps of degrees 2 and 3, respectively.

(a) Compute the homology groups of all the subcomplexes $A \subset X$ and the corresponding quotient complexes $X/A$.

(b) Show that the only subcomplex $A \subset X$ such that the quotient map $X \to X/A$ is a homotopy equivalence is the trivial subcomplex consisting of the 0-cell alone.

4. Suppose a simplicial complex structure on a closed surface of Euler characteristic $\chi$ has $v$ vertices, $e$ edges, and $f$ faces, which are triangles. Show that $e = \frac{3f}{2}$, $f = 2(v - \chi)$, $e = 3(v - \chi)$, and $e \leq v(v - 1)/2$. Deduce that $6(v - \chi) \leq v^2 - v$. For the torus conclude that $v \geq 7$, $f \geq 14$, and $e \geq 21$. Explain how the diagram at the right gives a simplicial complex structure on the torus realizing the minimum values $(v, e, f) = (7, 21, 14)$. For the projective plane show that $v \geq 6$, $f \geq 10$, and $e \geq 15$, and use the icosahedron to describe a simplicial structure realizing the minimum values $(v, e, f) = (6, 15, 10)$.

5. The degree of a homeomorphism $f: \mathbb{R}^n \to \mathbb{R}^n$ can be defined as the degree of the extension of $f$ to a homeomorphism of the one-point compactification $S^n$. Using this notion, fill in the details of the following argument which shows that $\mathbb{R}^n$ is not home-
omorphic to a product $X \times X$ if $n$ is odd. Assuming $\mathbb{R}^n = X \times X$, consider the homeomorphism $f$ of $\mathbb{R}^n \times \mathbb{R}^n = X \times X \times X \times X$ that cyclically permutes the factors, $f(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4, x_1)$. Then $f^2$ switches the two factors of $\mathbb{R}^n \times \mathbb{R}^n$, so $f^2$ has degree $-1$ if $n$ is odd. But $\deg(f^2) = (\deg f)^2 = +1$.

6. Using covering spaces and Proposition 1.32, do the following:
(a) Find all index 2 subgroups of a free group $F_2$ on 2 generators.
(b) Determine how many subgroups of $F_2$ there are of index 3. (You don’t have to list them explicitly.)

7. (a) Let $S^n$ be given the CW structure lifting the standard CW structure on $\mathbb{R}P^n$, so that $S^n$ has two $i$-cells for each $i \leq n$. Compute the resulting cellular chain complex for $S^n$ and verify that it has the correct homology groups. (Use orientations for the cells that lift orientations of the cells of $\mathbb{R}P^n$.)
(b) Compute the homology groups of the quotient space of $S^n$ obtained by identifying antipodal points in the standard $S^k \subset S^n$, for fixed $k < n$.

8. Generalizing Corollary 2B.4 on page 172 of the book, show that if a map from a compact manifold to a connected manifold of the same dimension is locally an embedding, then it is a covering space. (A manifold of dimension $n$ is by definition a Hausdorff space in which each point has an open neighborhood homeomorphic to $\mathbb{R}^n$.)

9. (a) Show that every map $X \to Y$ is homotopic to a constant map if $X$ is simply-connected and locally path-connected, and $Y$ is path-connected and has a contractible covering space.
(b) Let $T$ be the torus $S^1 \times S^1$. Show that every map $S^2 \to T$ is homotopic to a constant map, but there exist maps $T \to S^2$ that are not homotopic to constant maps.

10. Let $X$ be a finite connected graph having no vertex that is the endpoint of just one edge, and suppose that $H_1(X)$ is free abelian of rank $n > 1$, so the group of automorphisms of $H_1(X)$ is $GL_n(\mathbb{Z})$, the group of invertible $n \times n$ matrices with integer entries whose inverse matrix also has integer entries. Show that if $G$ is a finite group of homeomorphisms of $X$, then the homomorphism $G \to GL_n(\mathbb{Z})$ assigning to $g: X \to X$ the induced homomorphism $g_*: H_1(X) \to H_1(X)$ is injective. Show the same result holds if $H_1(X)$ is replaced by $H_1(X; \mathbb{Z}_m)$ with $m > 2$. What goes wrong when $m = 2$?