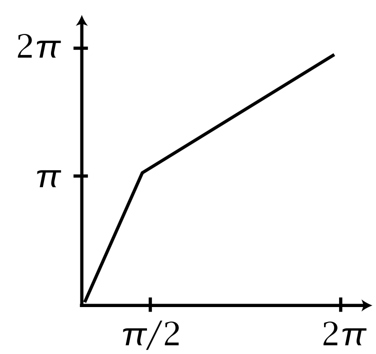
Math 453 Problem Set 4 — Solutions   Fall 2005

1. Suppose $f: X \to Y$ is continuous and one-to-one and Y is Hausdorff. Given two distinct points $x_1, x_2 \in X$, their images $f(x_1), f(x_2)$ are distinct points in Y having disjoint open neighborhoods U_1 and U_2, so $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are disjoint open neighborhoods of x_1 and x_2 in X. Thus X is Hausdorff.

2. If A and B are compact subspaces of a space X, then so is $A \cup B$ since if it is covered by open sets O_α then these sets cover each of A and B separately, so finitely many of the O_α's cover A and finitely many cover B, hence the union of these two finite subcovers is a finite subcover of $A \cup B$.
If in addition X is Hausdorff, then A and B are closed, hence also $A \cap B$. Then $A \cap B$ is closed in A so $A \cap B$ is compact since it is a closed subset of the compact space A.

3. For (x, y) to be in $\overline{A \times B}$ means that every neighborhood of (x, y) meets $A \times B$. It suffices to consider only basis neighborhoods $U \times V$ since every neighborhood contains such a neighborhood. For $U \times V$ to meet $A \times B$ is equivalent to U meeting A and V meeting B. This is equivalent to (x, y) being in $\overline{A} \times \overline{B}$. Thus $\overline{A \times B} = \overline{A} \times \overline{B}$.
For (x, y) to be in $\text{int}(A \times B)$ means that (x, y) has a neighborhood contained in $A \times B$. It suffices to consider only basis neighborhoods of the form $U \times V$, as before. For $U \times V$ to be contained in $A \times B$ is equivalent to $U \subset A$ and $V \subset B$. This is saying that (x, y) is in $\text{int}(A) \times \text{int}(B)$. Thus $\text{int}(A \times B) = \text{int}(A) \times \text{int}(B)$.

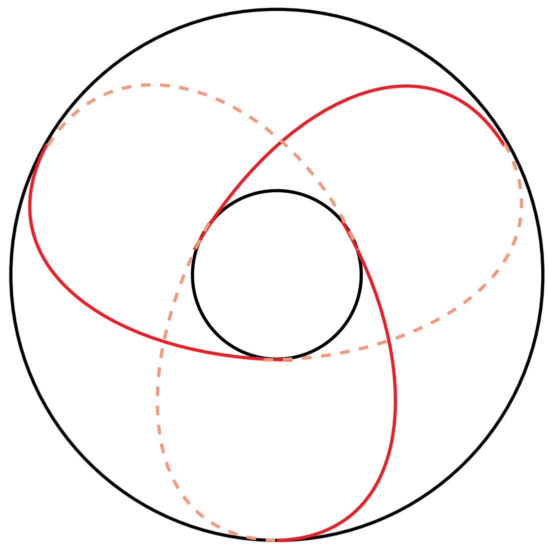
4. We know that $[0,1) \times [0,1)$ is not homeomorphic to $[0,1] \times [0,1]$ since it is not compact but $[0,1] \times [0,1]$ is compact. To show that $[0,1) \times [0,1)$ is homeomorphic to $[0,1] \times [0,1)$ we can replace the square by a disk, which is homeomorphic to a square by radial rescaling. For a disk the problem now is to find a homeomorphism from the disk to itself that takes an arc forming one quarter of its boundary to an arc forming one half of its boundary. In polar coordinates such a homeomorphism can be taken to be of the form $f(r, \theta) = (r, h(\theta))$ where h is a homeomorphism $[0, 2\pi] \to [0, 2\pi]$ taking $[0, \pi/2]$ to $[0, \pi]$. Such an h can be defined by a monotone function whose graph is shown at the right.



5. (a) The dot product of the ith and jth columns of an $n \times n$ matrix defines a function $f_{i,j}: \mathbb{R}^{n^2} \to \mathbb{R}$. This is continuous since it is defined by a polynomial. So $f_{i,j}^{-1}(c)$ is closed for each constant $c \in \mathbb{R}$. Let c be 0 if $i \neq j$ and 1 if $i = j$. Then the intersection of these closed sets $f_{i,j}^{-1}(c)$ as i and j go from 1 to n is $O(n)$, so $O(n)$ is closed in $\mathbb{R}^{n^2}$.

(b) Each entry of a matrix in $O(n)$ has absolute value at most 1, being a coordinate of a column which is a vector of length 1. Thus $O(n)$ lies inside the cube $[-1,1]\times\cdots\times[-1,1]$ in $\mathbb{R}^{n^2}$ so $O(n)$ is bounded as well as closed, hence $O(n)$ is compact.

6. The map $f:S^1\to S^1\times S^1$ given by $f(\theta)=(2\theta,3\theta)$ is continuous. It is one-to-one since in the first coordinate only θ and $\theta+\pi$ have the same image 2θ, but in the other coordinate θ and $\theta+\pi$ have distinct images 3θ and $3\theta+3\pi=3\theta+\pi$. Since f is a continuous one-to-one map from the compact space S^1 onto the space $f(S^1)$ which is Hausdorff since it is a subspace of a Hausdorff space, it follows that f is a homeomorphism. The image of f is a curve on the torus that starts and ends at the same point and wraps around the torus twice in the longitudinal direction (if we take the first coordinate of $S^1\times S^1$ to be the longitude) and three times in the meridional direction. This is a knot known as the trefoil knot, shown at the right.



7. (a) For a collection of subsets $O_\alpha\subset X$ we have $L_n\cap\bigcup_\alpha O_\alpha=\bigcup_\alpha(L_n\cap O_\alpha)$. So if each $L_n\cap O_\alpha$ is open in L_n, so will $L_n\cap\bigcup_\alpha O_\alpha$ be open in L_n. Thus the union of open sets in X is open. Finite intersections work in the same way, with $\bigcup$ replaced by $\bigcap$. Obviously X and $\varnothing$ also intersect each L_n in open sets in L_n. So we have a topology on X.

(b) Let L'_∞ be L_∞ with its left endpoint deleted. Then L'_∞ is open in the new topology on X since it intersects each L_n in an open set in L_n (the empty set unless $n=\infty$). But L'_∞ is not open in X if X is given the subspace topology, since every neighborhood of a point in L'_∞ contains points in other segments L_n.

(c) Let L'_n be L_n with its left endpoint deleted. Let X' be X with the right endpoint of each segment deleted. Then X' and all the L'_n's give an open cover of X with no finite subcover.