Math 453 Problem Set 2 — Solutions Fall 2005

1. For $f:X\to Y$ let $A = f(X)$. The open sets in A are the sets $A\cap O$ for O open in Y. We have $f^{-1}(O) = f^{-1}(A\cap O)$. Continuity of $f:X\to Y$ is equivalent to $f^{-1}(O)$ being open for all open sets O. Continuity of $f:X\to A$ is equivalent to $f^{-1}(A\cap O)$ being open for all open sets O. So the two versions of continuity are equivalent.

2. (a) The inclusion map $(0,1)\to\mathbb{R}$ is open since every open set in $(0,1)$ is open in $\mathbb{R}$, but it is not closed since $(0,1)$ is closed in $(0,1)$ but not in $\mathbb{R}$. A constant map $\mathbb{R}\to\mathbb{R}$ is obviously closed but not open.

(b) The projection map $p:\mathbb{R}^2\to\mathbb{R}$, $p(x,y) = x$ is open since if O is open in $\mathbb{R}^2$ then $p(O)$ is open in $\mathbb{R}$ because each $x\in p(O)$ is the projection of some point $(x,y)\in O$, and an open disk in O containing (x,y) projects to an open interval in $p(O)$ containing x. The map p is not closed since the graph of the function $y = 1/x$ for $x > 0$ is closed in $\mathbb{R}^2$ but its projection is the interval $(0,\infty)$ which is not closed in $\mathbb{R}$.

(c) The map $f:\mathbb{R}\to S^1$, $f(x) = (\cos x, \sin x)$ is open since it sends each open interval in $\mathbb{R}$ to either an open arc in S^1 or all of S^1. However, f is not closed since it sends the closed set $S = \{\,2n\pi + 1/n \mid n = 1,2,\cdots\}$ to a sequence of points in S^1 converging to $(1,0)$, but $(1,0)\notin f(S)$.

3. The map $\mathbb{R}^2\to\mathbb{R}$ sending (x,y) to $x+y$ is continuous because the inverse image of an open interval (a,b) is the open region between two parallel lines $x+y = a$ and $x+y = b$. For the function xy the inverse image of (a,b) is the open region between two hyperbolas $xy = a$ and $xy = b$. Here there are a couple cases to consider depending on the signs of a and b, including the degenerate case $xy = 0$ where the hyperbola becomes the coordinate axes.

4. For an open set $O\subset Y$, the hypothesis that the restriction of f to O_α is continuous guarantees that $f^{-1}(O)\cap O_\alpha$ is open in O_α, hence equals $O_\alpha\cap U_\alpha$ for some open set U_α in X. This implies that $f^{-1}(O)\cap O_\alpha$ is open in X since it is the intersection of two open sets in X. The union $\bigcup_\alpha(f^{-1}(O)\cap O_\alpha)$ is $f^{-1}(O)$ since $X = \bigcup_\alpha O_\alpha$, so $f^{-1}(O)$ is open, and f is continuous.

5. (a) Every neighborhood in $\mathbb{R}_h$ of a point x contains one of the special neighborhoods $[x, x+1/n)$. Hence x is a limit point of A if and only if each neighborhood $[x, x+1/n)$ contains a point $x_n\in A$. This implies that $x_n\ge x$ and $|x_n - x|\to 0$. Conversely, if A contains such a sequence then every neighborhood $[x, x+1/n)$ meets A so x is a limit point of A.

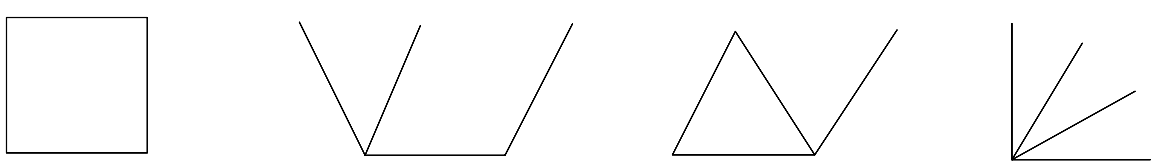
(b) For $f : \mathbb{R}_h \to \mathbb{R}$ to be continuous at a point x means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $f([x, x+\delta)) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$. This is what f being continuous from the right means. [You should give a few more details than this.]

6. If D_1 and D_2 are open disks in $\mathbb{R}^2$ whose closures $\overline{D}_1$ and $\overline{D}_2$ intersect in exactly one point, then: (a) $D_1 \cup D_2$ is not connected since each of D_1 and D_2 is open in $D_1 \cup D_2$. (b) $\overline{D}_1 \cup \overline{D}_2$ is connected since it is path-connected — any two points can be joined to the point $\overline{D}_1 \cap \overline{D}_2$ by straight line segments. (c) The same reasoning shows $\overline{D}_1 \cup D_2$ is connected.

7. Suppose $\overline{A}$ is the disjoint union of subsets B and C which are closed in $\overline{A}$ and hence also closed in X since $\overline{A}$ is closed in X. Then $A \cap B$ and $A \cap C$ are disjoint closed sets in A whose union is A, so one of these two sets must equal A, say $A \cap B = A$. This says $A \subset B$, so $\overline{A} \subset \overline{B} = B$. Since we originally had $\overline{A} = B \amalg C$ this implies $B = \overline{A}$ and $C = \varnothing$.

8. The subspace $X \subset \mathbb{R}^2$ consisting of points (x, y) such that at least one of x and y is rational is connected since it is path-connected. Namely, there is a path in X from any point (x, y) to the origin by first going either horizontally (if y is rational) or vertically (if x is rational) to one of the coordinate axes, then going along that coordinate axis to the origin.

9. The graph in (a) has no cutpoints, but the other three graphs do, so (a) is not homeomorphic to any of the other graphs. The number of non-cutpoints in (b), (c), and (d) is 3, ∞, and 4 respectively since in (b) and (d) only the 'free' endpoints of the segments are non-cutpoints, while in (c) all the points in the triangle except the right-most point are non-cutpoints. So no two of the graphs are homeomorphic.



(a) (b) (c) (d)