1. Let O be a nonempty open set in $\mathbb{R}$, and let x be a point in O. Since O is open, there is an open interval in O containing x. Let O_x be the union of all the open intervals in O that contain x. Then O_x is open since it is a union of open sets. Also, O_x has to be an interval, since if y is any point in O_x other than x then the whole interval between x and y must lie in O_x because, by the definition of O_x, there is some open interval in O containing both x and y, hence containing all points between x and y. Thus $O_x = (a_x, b_x)$ where a_x could be $-\infty$ and b_x could be $+\infty$. Consider now two of these intervals O_x and O_y. We claim that if they intersect, then they have to be equal. For suppose they intersect. Then their union $O_x \cup O_y$ will also be an open interval in O containing x, so since O_x was the union of all such intervals we must have $O_x \cup O_y \subset O_x$ and hence $O_x \cup O_y = O_x$. The same argument shows that $O_x \cup O_y = O_y$, so we have $O_x = O_y$ as claimed. Now we have O equal to the union of all these intervals O_x with the property any two of them are either equal or are disjoint. Thus O is the disjoint union of all the intervals O_x that are distinct.

2. (a) If O is an open set contained in A then for each $x \in O$ we have $x \in O \subset A$, which is exactly the condition for x to be in $\text{int}(A)$. Thus any point $x \in O$ is in $\text{int}(A)$, so $O \subset \text{int}(A)$. So $\text{int}(A)$ is an open set contained in A which contains every other open set contained in A. This makes $\text{int}(A)$ the largest open set contained in A.

(b) Let C be a closed set containing A. If x is a point in $\overline{A}$ then every neighborhood of x meets A and hence also meets C since $C \supset A$. Thus x is in $\overline{C}$, so $\overline{A} \subset \overline{C}$. But $\overline{C} = C$ so $\overline{A} \subset C$. Thus $\overline{A}$ is a closed set containing A which is contained in every other closed set containing A, hence $\overline{A}$ is the smallest closed set containing A.

3. Let $\mathcal{O}$ be the collection of all intervals $I_a = (a, \infty)$ in $\mathbb{R}$, including the cases $I_\infty = \varnothing$ and $I_{-\infty} = \mathbb{R}$. Show that $\mathcal{O}$ defines a topology on $\mathbb{R}$. In this topology, what is the closure of a set $A \subset \mathbb{R}$?

We have $\varnothing \in \mathcal{O}$ and $\mathbb{R} \in \mathcal{O}$. For a collection of intervals $I_{a_\alpha} = (a_\alpha, \infty)$, their union equals I_b where b is the greatest lower bound of the numbers a_α if these numbers have a lower bound, and otherwise $b = -\infty$. So the union of a collection of intervals in $\mathcal{O}$ is in $\mathcal{O}$. The intersection of a finite number of intervals I_{a_α} is I_b where b is the largest of the numbers a_α, so this intersection is again in $\mathcal{O}$. Thus the axioms for a topology are satisfied. The closed sets are the complements of the open sets, in other words the intervals $(-\infty, a]$, $(-\infty, \infty)$, and $\varnothing$. The closure of a set A is the smallest closed set containing A. If A is nonempty and bounded above, its closure is the interval $(-\infty, a]$ where a is the least upper bound for the elements of A. If A is not bounded above then $\overline{A} = (-\infty, \infty)$. If $A = \varnothing$ then $\overline{A} = \varnothing$.

4. (a) For a point x to be in $\overline{X - A}$ means that every neighborhood of x meets $X - A$, or in other words, no neighborhood of x lies entirely in A. This means exactly that x is not in $\mathrm{int}(A)$. Thus $\overline{X - A} = X - \mathrm{int}(A)$.
(b) Replacing A by $X - A$ in part (a) gives the equation $\overline{A} = X - \mathrm{int}(X - A)$. Taking the complements of both sides gives $X - \overline{A} = \mathrm{int}(X - A)$.

5. We can use the fact that if $A \subset B$ then $\overline{A} \subset \overline{B}$ (which holds since if every neighborhood of x meets A then it meets B) and $\mathrm{int}(A) \subset \mathrm{int}(B)$ (which holds since if x has a neighborhood contained in A then it has a neighborhood contained in B).
(a) Since $A \cup B$ contains A and B, $\overline{A \cup B}$ contains $\overline{A}$ and $\overline{B}$ and hence their union. To obtain the opposite inclusion $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$, suppose x is a limit point of $A \cup B$. If x is not a limit point of either A or B separately, then x has a neighborhood disjoint from A and another neighborhood disjoint from B. The intersection of these two neighborhoods would be a neighborhood disjoint from $A \cup B$, contradicting the assumption that x was a limit point of $A \cup B$. Thus x must be a limit point of either A or B (or both). This shows $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.
(b) $A \cap B \subset A$ and $A \cap B \subset B$ imply that $\overline{A \cap B} \subset \overline{A}$ and $\overline{A \cap B} \subset \overline{B}$ so $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. For an example where equality does not hold, take $X = \mathbb{R}$, $A = (0, 1)$, and $B = (1, 2)$, so $\overline{A \cap B} = \varnothing$ but $\overline{A} \cap \overline{B} = \{1\}$. Another example would be $X = \mathbb{R}$, $A = \mathbb{Q}$, $B = \mathbb{R} - \mathbb{Q}$.
(c) Since $A \cap B \subset A$ and $A \cap B \subset B$ we have $\mathrm{int}(A \cap B) \subset \mathrm{int}(A)$ and $\mathrm{int}(A \cap B) \subset \mathrm{int}(B)$ hence $\mathrm{int}(A \cap B) \subset \mathrm{int}(A) \cap \mathrm{int}(B)$. For the opposite inclusion, if $x \in \mathrm{int}(A) \cap \mathrm{int}(B)$ then x has a neighborhood contained in A and a neighborhood contained in B, so the intersection of these two neighborhoods is a neighborhood contained in $A \cap B$, hence $x \in \mathrm{int}(A \cap B)$.
(d) $\mathrm{int}(A) \cup \mathrm{int}(B) \subset \mathrm{int}(A \cup B)$ since $A \subset A \cup B$ implies $\mathrm{int}(A) \subset \mathrm{int}(A \cup B)$ and $B \subset A \cup B$ implies $\mathrm{int}(B) \subset \mathrm{int}(A \cup B)$. An example where equality fails is $X = \mathbb{R}$ with $A = [0, 1]$ and $B = [1, 2]$, or one could take $A = \mathbb{Q}$ and $B = \mathbb{R} - \mathbb{Q}$.

6. We know that X is the disjoint union of $\mathrm{int}(A)$, ∂A, and $\mathrm{int}(X - A)$, so in order to show that ∂A contains $\partial(\mathrm{int}(A))$ it will suffice to show that $\partial(\mathrm{int}(A))$ is disjoint from $\mathrm{int}(A)$ and $\mathrm{int}(X - A)$. To do this, suppose x is a point in $\mathrm{int}(A)$, so x has an open neighborhood $O \subset A$. Then $O \subset \mathrm{int}(A)$ since $\mathrm{int}(A)$ is the largest open set contained in A. Thus x has a neighborhood contained in $\mathrm{int}(A)$ so x is not in $\partial(\mathrm{int}(A))$ since every every neighborhood of a point in $\partial(\mathrm{int}(A))$ must meet both $\mathrm{int}(A)$ and the complement of $\text{int}(A)$. This shows that $\partial(\text{int}(A))$ is disjoint from $\text{int}(A)$. To show it is also disjoint from $\text{int}(X - A)$ let x be a point of $\text{int}(X - A)$, so x has an open neighborhood $O \subset X - A$. This O is disjoint from $\text{int}(A)$ so x in not in $\partial(\text{int}(A))$ since it has a neighborhood that does not meet both $\text{int}(A)$ and the complement of $\text{int}(A)$.

How does $\partial(A \cup B)$ relate to ∂A and ∂B? The example $X = \mathbb{R}$, $A = [0,1]$, $B = [1,2]$ shows that the only possible relation is $\partial(A \cup B) \subset \partial A \cup \partial B$, and that equality does not always hold. To show that the inclusion does hold, let x be a point in $\partial(A \cup B)$. Then every neighborhood O of x meets $A \cup B$ and the complement of $A \cup B$, so in particular O meets the complements of both A and B. Since O meets either A or B and the complement of each of them, x must lie in either ∂A or ∂B.

7. If Y is a subspace of X and Z is a subspace of Y, then the open sets in Z have the form $Z \cap (Y \cap O)$ for open O in X. We have $Z \cap (Y \cap O) = Z \cap O$ since $Z \subset Y$, so these sets are exactly the open sets $Z \cap O$ in Z regarded as a subspace of X.

8. First a preliminary observation: The set of points in A of distance less than ε from x is the intersection $A \cap D_\varepsilon(x)$ where $D_\varepsilon(x)$ is the open disk of radius ε centered at x. Now suppose that a subset O of A has the property that for each $x \in O$ there is a disk $D_\varepsilon(x)$ with $A \cap D_e(x) \subset O$. Then O is open in A since each of its points has an open neighborhood $A \cap D_\varepsilon(x)$ contained in O. Conversely, if O is open in A then $O = A \cap O'$ for some open set O' in $\mathbb{R}^2$. This O' is a union of open disks D_α since such disks form a basis for the topology on $\mathbb{R}^2$. For each $x \in D_\alpha$ the disk $D_\varepsilon(x)$ will be contained in D_α if ε is sufficiently small. Thus for each $x \in O$ there is a disk $D_\varepsilon(x)$ contained in O', hence $A \cap D_\varepsilon(x)$ is contained in O.

9. If x is a point in $\text{int}_X(A)$ this means there is an open set O in X with $x \in O \subset A$. Then we certainly have $x \in O \cap Y \subset A$ since $O \subset A \subset Y$. Thus x has an open neighborhood $O \cap Y$ in A with the subspace topology from Y. Hence $x \in \text{int}_Y(A)$, which shows that $\text{int}_X(A) \subset \text{int}_Y(A)$. For an example where equality does not hold, take $X = \mathbb{R}^2$ and $Y = \mathbb{R}$ viewed as a subspace of $\mathbb{R}^2$, the x-axis, and let $A = [0,1] \subset \mathbb{R} = Y$. Then $\text{int}_X(A) = \varnothing$ and $\text{int}_Y(A) = (0,1)$.

10. If A is open in Y then $A = Y \cap O$ for some open set O in X. If Y is open in X then A is open in X since it is the intersection of the two opens sets O and Y in X. The same argument works if ‘open’ is replaced by ‘closed’ throughout.