Math 453 Prelim 2 **Fall 2005**

Rules: Once you look at the problems on the exam, the only sources you can consult are your notes from class, the online notes on the course webpage, and your homework papers and the posted solutions. You can also send me email if you have questions about the meaning of the problems or about material covered in the course.

1. (a) Show that a continuous map $f:X\to Y$ that is onto and open (i.e., f takes open sets to open sets) is a quotient map.
(b) Determine the quotient space of $\mathbb{R}$ with the identifications $x \sim x+n$ for all $x\in\mathbb{R}$ and $n\in\mathbb{Z}$. (And prove that your answer is correct.)
(c) Determine the quotient space of $\mathbb{R}^2$ with the identifications $(x,y)\sim(x+m,y+n)$ for all $x,y\in\mathbb{R}$ and $m,n\in\mathbb{Z}$. (And prove that your answer is correct.)
(d) Determine the quotient space of $\mathbb{R}^2$ with the identifications $(x,y)\sim((-1)^n x+m, y+n)$ for all $x,y\in\mathbb{R}$ and $m,n\in\mathbb{Z}$. (And prove that your answer is correct.)

2. (a) Embed the torus T^2 in $\mathbb{R}^3$ by rotating the circle $(y-2)^2+z^2=1$ in the yz-plane about the z-axis. The antipodal map $\mathbb{R}^3\to\mathbb{R}^3$, $x\mapsto -x$, then takes T^2 to itself. Prove that the quotient space $T^2/(x\sim -x)$ is homeomorphic to the Klein bottle. (This is analogous to the projective plane being the quotient $S^2/(x\sim -x)$.)
(b) Generalize part (a) to the n-hole torus T_n, i.e., describe an embedding of T_n in $\mathbb{R}^3-\{0\}$ that is symmetric with respect to the map $x\mapsto -x$, and determine the quotient space $T_n/(x\sim -x)$.

3. Let L be the set of all lines in $\mathbb{R}^2$, solutions of an equation $ax+by=c$. We can define a topology on L in two ways: First as a quotient space of the space X of all pairs (v,w) of points $v,w\in\mathbb{R}^2$ with $v\neq w$ (so X is an open set in $\mathbb{R}^4=\mathbb{R}^2\times\mathbb{R}^2$) under the function $X\to L$ sending (v,w) to the line through v and w. And second as a quotient space of $S^1\times\mathbb{R}$ under the function $S^1\times\mathbb{R}\to L$ sending (u,t) to the line perpendicular to the unit vector u and passing through the point tu (the scalar multiple of u by the real number t).
(a) Show these two topologies on L are the same.
(b) Show that L with this topology is homeomorphic to a familiar space we have seen often in the course (more precisely, L is homeomorphic to this familiar space with its boundary removed). Hint: Determine how the map $S^1\times\mathbb{R}\to L$ fails to be one-to-one.

4. Let $f:X\to Y$ be a quotient map. Define the *saturation* of a set $A\subset X$ to be the set $f^{-1}(f(A))$, all the points in X that have the same image under f as points in A.
(a) Show that f is a closed map (i.e., f takes closed sets to closed sets) if and only if the saturation of each closed set in X is closed.
(b) Show that if f is a closed map and X is a normal space, then so is Y. (Recall that a normal space is one that is Hausdorff and has the property that any two disjoint closed sets have disjoint open neighborhoods.)
(c) Give an example of a quotient map between normal spaces that is not a closed map.