Math 453 Prelim 1 — Solutions Fall 2005

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1. Show that the only continuous maps $f : \mathbb{R} \to \mathbb{Q}$ are the constant functions.

We can view f as a map $\mathbb{R} \to \mathbb{R}$. Since $\mathbb{R}$ is connected, so is $f(\mathbb{R})$. The only connected sets in $\mathbb{R}$ are intervals. But $f(\mathbb{R})$ is contained in $\mathbb{Q}$ and the only intervals contained in $\mathbb{Q}$ are the intervals $[a,a]$, so $f(\mathbb{R})$ must be such an interval, ie, a constant.

2. (a) Show $X \times Y$ is connected if and only if both X and Y are connected. (b) Do the same for 'path-connected'.

For the easier direction, the 'only if' statements, we know the projections of $X \times Y$ onto X and Y are continuous, and they are onto, so if $X \times Y$ is connected so are X and Y since they are the images of a connected space under a continuous map. The same reasoning holds with 'path-connected' in place of 'connected'.

For the converse statement in (a), suppose X and Y are connected. Let U be a nonempty set in $X \times Y$ which is both open and closed. Suppose (a,b) is any point in U. Then U intersects $\{a\} \times Y$ in an open and closed set in $\{a\} \times Y$. Since $\{a\} \times Y$ is connected (being homeomorphic to Y) it follows that U must contain all of $\{a\} \times Y$. The same reasoning shows U must contain $X \times \{b\}$ whenever it contains a point (a,b). Thus if we start with a point (a,b) in U we conclude that U contains all points (a,y) and therefore also all points (x,y), so $U = X \times Y$. Hence $X \times Y$ is connected.

For the other half of (b), suppose X and Y are path-connected. Given two points (a,b) and (c,d) in $X \times Y$, there is a path f in X from a to c and a path g in Y from b to d. Then the function $h(t) = (f(t), g(t))$ gives a path in $X \times Y$ from (a,b) to (c,d). Continuity of h follows from continuity of f and g.

3. Show that if X is Hausdorff and $x_1, \cdots, x_n$ are distinct points in X, then there are disjoint open sets $U_1, \cdots, U_n$ in X with $x_i \in U_i$ for each i.

For each set $\{x_i, x_j\}$ with $i \neq j$ there are two disjoint open neighborhoods U_{ij} of x_i and U_{ji} of x_j. Let U_i be the intersection of all the sets U_{ij} with $j \neq i$. This is open since it is a finite intersection of open sets, and it contains x_i since each U_{ij} contains x_i. And we have $U_i \cap U_j = \emptyset$ if $i \neq j$ since $U_i \cap U_j$ is contained in $U_{ij} \cap U_{ji} = \emptyset$.

4. Suppose we are given two continuous maps $f, g : X \to Y$. Show that if Y is Hausdorff then the set $A = \{\, x \mid f(x) = g(x) \,\}$ is closed in X.

We will show that $X - A$ is open. If $x \in X - A$ then $f(x) \neq g(x)$, so since Y is Hausdorff there exist disjoint open neighborhoods U and V of $f(x)$ and $g(x)$. Let $W = f^{-1}(U) \cap g^{-1}(V)$, so W is an open neighborhood of x with $f(W) \subset U$ and

$g(W) \subset V$. Since $U \cap V = \varnothing$ this means that $f \neq g$ at all points of W, hence $W \subset X - A$. This shows $X - A$ is open.

5. Suppose that in $X \times Y$ we have an open set W that contains the product $A \times B$ of compact subspaces $A \subset X$ and $B \subset Y$. Show that there are open sets $U \subset X$ and $V \subset Y$ with $A \times B \subset U \times V \subset W$.

For each $(a,b) \in A \times B$ we can choose a basic open set $U_{ab} \times V_{ab} \subset W$ containing (a,b) since W is open. The sets V_{ab} with a fixed and b varying form an open cover of B, so since B is compact there is a finite subcover, call it $V_{a1}, \cdots, V_{an}$, and let U_a be the intersection of the corresponding finite collection of sets U_{ab}, so U_a is an open set containing a. We have $U_a \times V_{ai} \subset W$ for each i, so if we let $V_a = \bigcup_i V_{ai}$ we have $B \subset V_a$ and $U_a \times V_a \subset W$. Now we let a vary over A. The sets U_a form an open cover of A, so since A is compact this cover has a finite subcover. Call this subcover $U_1, \cdots, U_m$ and let $V_1, \cdots, V_m$ be the corresponding sets V_a. Let $U = \bigcup_i U_i$ and $V = \bigcap_i V_i$. Since the U_i's cover A we have $A \subset U$. Since $B \subset V_i$ for each i we have $B \subset V$. Also we had $U_a \times V_a \subset W$ so $U_i \times V_i \subset W$ hence $U_i \times V \subset W$ and therefore also $U \times V \subset W$.

For an example where the conclusion is no longer true if A is not compact, let $X = Y = \mathbb{R}$ with $A \times B = \mathbb{R} \times \{0\}$ and W the region between the graphs of $y = 1/(1+x^2)$ and $y = -1/(1+x^2)$. We would then have to have $U = \mathbb{R}$, but W contains no product $\mathbb{R} \times V$ with V an open neighborhood of 0 in $\mathbb{R}$.

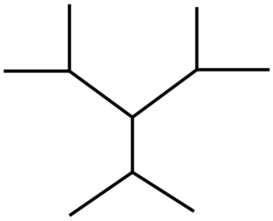
6. (a) Show that for any continuous map $f:[-1,1] \to [-1,1]$ there exists a point $x \in [-1,1]$ with $f(x) = x$.

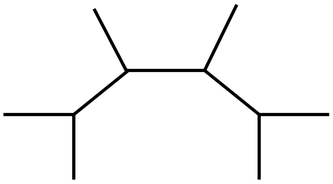
Consider the function $g(x) = f(x) - x$, which is continuous since f is continuous. We want to show that $g(x) = 0$ for some x. We have $g(1) \leq 0$ since $f(1) \leq 1$. Similarly $g(-1) \geq 0$ since $f(-1) \geq -1$. Since $[-1,1]$ is connected, so is $g([-1,1])$, so $g([-1,1])$ is an interval, and it contains a value $g(-1) \geq 0$ and a value $g(1) \leq 0$ so it must also contain the value 0. (You could also just quote the intermediate value theorem.)

(b) Show that for any continuous map $f:S^1 \to \mathbb{R}$ there exists a point $x \in S^1$ with $f(x) = f(-x)$.

Consider the function $g(x) = f(x) - f(-x)$, which is continuous since f is continuous. We want to show that $g(x) = 0$ for some x. The function g satisfies $g(-x) = -g(x)$, so either g is identically 0, in which case we're done, or g takes on both positive and negative values. Since S^1 is connected (being path-connected) so is $g(S^1)$, so $g(S^1)$ is an interval, and this interval contains both positive and negative values, hence also 0. [This proof didn't follow the hint to consider the values $g(x)$ as x goes halfway around the circle. If we do this, g would be a function from a semicircle to $\mathbb{R}$ taking values $g(x)$ and $g(-x) = -g(x)$ at the two endpoints of the semicircle, so the same reasoning would still apply.]

7. Use cutpoints to show that the following two graphs in $\mathbb{R}^2$ are not homeomorphic:





First proof: Each graph has exactly 4 special cutpoints whose complement has 3 components. In the first graph one of these special cutpoints (the one in the center) has one special cutpoint in each of the three components of its complement. In the other graph none of the four special cutpoints has a special cutpoint in each of the three components of its complement. So the two graphs are not homeomorphic. (Any homeomorphism must take special cutpoints to special cutpoints.)

Second proof: Again consider the four special cutpoints, but consider also the six non-cutpoints in each graph. In the first graph there is a special cutpoint, the one in the middle, that has two of the six non-cutpoints in each of the three components of its complement. In the second graph none of the special cutpoints has this property.