Math 453 Problem Set 5 — Fall 2005

1. For a set X define $d: X \times X \to \mathbb{R}$ by setting $d(x,x) = 0$ and $d(x,y) = 1$ if $x \neq y$. Show that d is a metric. What is the topology defined by this metric?

2. (a) Given a metric d on a set X, show that each of the functions $d'(x,y)$ listed below is again a metric, and that d and d' define the same topology on X. Note that in (b) and (c) the values of d' are bounded above by 1.

(a) $d'(x,y) = kd(x,y)$ for any constant $k > 0$.

(b) $d'(x,y) = \min\{d(x,y), 1\}$.

(c) $d'(x,y) = d(x,y)/(1 + d(x,y))$.

3. For a metric space X defined by a metric d show that the function $d: X \times X \to \mathbb{R}$ is continuous.

4. Let X be a metric space with metric d. For subsets $A, B \subset X$ define $d(A,B) = \inf\{\, d(a,b) \mid a \in A \text{ and } b \in B \,\}$. Show that if A and B are compact then there exist points $a \in A$ and $b \in B$ with $d(a,b) = d(A,B)$, so in particular $d(A,B) > 0$ if $A \cap B = \emptyset$. Show by an example that both these statements can be false if A and B are only assumed to be closed instead of compact.

5. Let X be the subset of $\mathbb{R}^2$ which is the union of the line segments L_n from $(0,0)$ to $(1, 1/n)$ for $n = 1, 2, \cdots$, together with the limiting segment L_∞ from $(0,0)$ to $(1,0)$. Define a topology $\mathcal{O}$ on X by saying that a set $O \subset X$ is in $\mathcal{O}$ if $O \cap L_n$ is open in L_n for each $n \leq \infty$, where L_n is given the subspace topology from $\mathbb{R}^2$. Show that this topology on X is normal but is not defined by any metric on X. Hint: Given a metric on X, find a sequence of points $x_n \in L_n$, $n = 1, 2, \cdots$, converging to $(0,0)$ in the topology defined by the metric but not in the topology $\mathcal{O}$.

For the next problems recall that a map $f: X \to Y$ is a quotient map if it is onto and has the property that a set $O \subset Y$ is open if and only if $f^{-1}(O)$ is open in X.

6. Show that a continuous map $f: X \to Y$ is a quotient map if there exists a continuous map $g: Y \to X$ such that the composition $fg: Y \to Y$ is the identity map. Apply this to show that a projection map $X \times Y \to X$, $(x,y) \mapsto x$, is a quotient map.

7. Show that the composition of two quotient maps is a quotient map.

8. Show that there exist quotient maps $D^2 \to S^1$, $S^2 \to D^2$, $S^2 \to S^1$, $S^2 \to S^1 \times S^1$, and $S^1 \times S^1 \to S^2$. [You do not need to give formulas for these maps, a precise geometric description will be sufficient. Careful proofs of continuity are not required, but the quotient maps do have to be continuous. Remember that $S^1 \times S^1$ is a torus.]

9. (a) Show that there is a quotient map $(0,1) \to [0,1]$, but not $[0,1] \to (0,1)$. (b) If C is the Cantor set, show there is a quotient map $C \to [0,1]$ but not $[0,1] \to C$.