Math 453 Problem Set 4 **Fall 2005**

1. Show that if $f: X \to Y$ is continuous and one-to-one and if Y is Hausdorff then so is X.

2. Show that if A and B are compact subspaces of a space X, then so is $A \cup B$. If in addition X is Hausdorff, show that $A \cap B$ is compact.

3. For subspaces $A \subset X$ and $B \subset Y$ show that $\overline{A \times B} = \overline{A} \times \overline{B}$ and $\mathrm{int}(A \times B) = \mathrm{int}(A) \times \mathrm{int}(B)$.

4. Show that $[0,1) \times [0,1)$ is homeomorphic to $[0,1] \times [0,1)$ but not to $[0,1] \times [0,1]$.

5. Consider the orthogonal group $O(n)$ consisting of all $n \times n$ orthogonal matrices, i.e., $n \times n$ matrices whose columns form an orthonormal basis $v_1, \cdots, v_n$ for $\mathbb{R}^n$. We put a topology on $O(n)$ by regarding it as a subspace of $\mathbb{R}^{n^2}$, taking the n^2 entries of a matrix in $O(n)$ as the n^2 coordinates of a point in $\mathbb{R}^{n^2}$. (a) Show that $O(n)$ is a closed subset of $\mathbb{R}^{n^2}$ by considering the dot products $v_i \cdot v_j$ of the columns of matrices in $O(n)$ as functions $\mathbb{R}^{n^2} \to \mathbb{R}$. (b) Show that $O(n)$ is compact.

6. Consider the function $f: S^1 \to S^1 \times S^1$ given by $f(\theta) = (2\theta, 3\theta)$ where we think of points in S^1 as angles θ. Show that f is a homeomorphism onto its image, and draw a picture of this image on the torus $S^1 \times S^1$.

7. Let X be the subset of $\mathbb{R}^2$ which is the union of the line segments L_n from $(0,0)$ to $(1, 1/n)$ for $n = 1, 2, \cdots$, together with the limiting segment L_∞ from $(0,0)$ to $(1,0)$. Define a topology on X by saying that a set $O \subset X$ is open if $O \cap L_n$ is open in L_n for all n, including $n = \infty$. (Here L_n is given the subspace topology from $\mathbb{R}^2$.) (a) Show the axioms for a topology are satisfied. (b) Find a set that is open in this topology but not in the topology on X as a subspace of $\mathbb{R}^2$. (c) Show that X is not compact in this new topology, in contrast with the subspace topology where it is compact since it is a closed bounded subset of $\mathbb{R}^2$.