Math 453 Problem Set 2 — Fall 2005

1. For a function $f:X\to Y$ the image $f(X)=\{f(x)\mid |x\in X\}$ is a subspace of Y. Show that $f:X\to Y$ is continuous if and only if $f:X\to f(X)$ is continuous.

2. A map $f:X\to Y$ is said to be open if $f(O)$ is open in Y whenever O is open in X. Similarly, $f:X\to Y$ is said to be closed if $f(C)$ is closed in Y whenever C is closed in X. (a) Give an example of a map that is open but not closed, and an example of a map that is closed but not open. (b) Determine whether the projection map $\mathbb{R}^2\to\mathbb{R}$ sending (x,y) to x is open or closed. (c) Do the same for the map $f:\mathbb{R}\to S^1$, $f(x)=(\cos x,\sin x)$, where S^1 is the unit circle $x^2+y^2=1$ in $\mathbb{R}^2$.

3. Show the two maps $\mathbb{R}^2\to\mathbb{R}$ sending (x,y) to $x+y$ and xy are continuous, using only definitions and results from this class, not results from calculus for example.

4. Suppose a space X is the union of a collection of open subsets O_α. Show that a map $f:X\to Y$ is continuous if its restriction to each subspace O_α is continuous.

5. Let $\mathbb{R}_h$ denote $\mathbb{R}$ with the 'half-open interval topology' having as basis the intervals $[a,b)$. (a) For a subset $A\subset\mathbb{R}_h$ show that a point x lies in the closure of A if and only if there is a sequence $\{x_n\}$ in A such that $x_n\ge x$ and $|x_n-x|\to 0$. (b) Show that a function $f:\mathbb{R}_h\to\mathbb{R}$ (with the usual topology on $\mathbb{R}$) is continuous if and only if it is continuous from the right at each point x, that is, $\lim f(x+\varepsilon)=f(x)$ where the limit is as $\varepsilon\to 0$ with $\varepsilon>0$.

6. Let D_1 and D_2 be two open disks in $\mathbb{R}^2$ whose closures $\overline{D}_1$ and $\overline{D}_2$ intersect in exactly one point, so the boundary circles of the two disks are tangent. Determine which of the following subspaces of $\mathbb{R}^2$ are connected: (a) $D_1\cup D_2$. (b) $\overline{D}_1\cup\overline{D}_2$. (c) $\overline{D}_1\cup D_2$.

7. Show that if a subspace A of a space X is connected then so is its closure $\overline{A}$.

8. Show the subspace $X\subset\mathbb{R}^2$ consisting of points (x,y) such that at least one of x and y is rational is connected.

9. By counting cut points and non-cut points show that no two of the following four graphs in $\mathbb{R}^2$ are homeomorphic. (Each graph is a closed subspace of $\mathbb{R}^2$, the union of a finite number of closed line segments.)