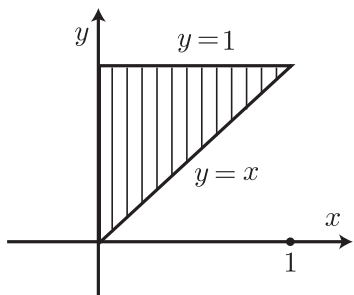


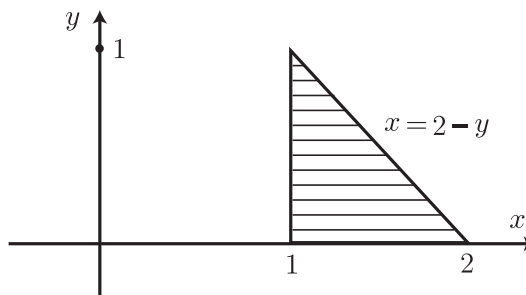
Problems from the book:

Section 5.4:

$$\begin{aligned}
 \mathbf{1a.} \quad \int_0^1 \int_x^1 xy \, dy \, dx &= \int_0^1 \left[\frac{x^2 y}{2} \Big|_x^1 \right] dx = \int_0^1 \left(\frac{x}{2} - \frac{x^3}{2} \right) dx = \left(\frac{x^2}{4} - \frac{x^4}{8} \right) \Big|_0^1 = \frac{1}{8} \\
 \int_0^1 \int_0^y xy \, dx \, dy &= \int_0^1 \left[\frac{xy^2}{2} \Big|_0^y \right] dy = \int_0^1 \frac{y^3}{2} dy = \frac{y^4}{8} \Big|_0^1 = \frac{1}{8}
 \end{aligned}$$



(a)



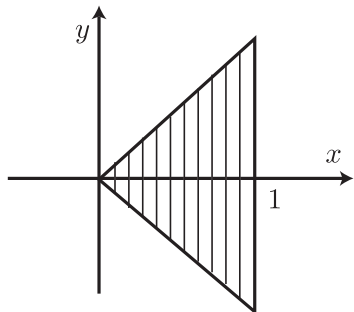
(c)

$$\begin{aligned}
 \mathbf{1c.} \quad \int_0^1 \int_1^{2-y} (x+y)^2 \, dx \, dy &= \int_0^1 \left[\frac{(x+y)^3}{3} \Big|_1^{2-y} \right] dy = \int_0^1 \left(\frac{8}{3} - \frac{(1+y)^3}{3} \right) dy \\
 &= \left(\frac{8y}{3} - \frac{(1+y)^4}{12} \right) \Big|_0^1 = \frac{8}{3} - \frac{16}{12} + \frac{1}{12} = \frac{17}{12}
 \end{aligned}$$

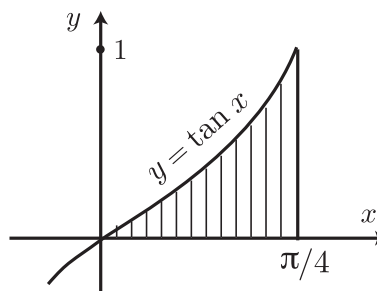
$$\int_1^2 \int_0^{2-x} (x+y)^2 \, dy \, dx = \int_1^2 \left[\frac{(x+y)^3}{3} \Big|_0^{2-x} \right] dx = \int_1^2 \left(\frac{8}{3} - \frac{x}{3} \right) dx = \left(\frac{8x}{3} - \frac{x^2}{6} \right) \Big|_1^2 = \frac{17}{12}$$

2a. For $\int_{-1}^1 \int_{|y|}^1 (x+y)^2 \, dx \, dy$ the region of integration is as shown. We can compute the integral by reversing the order of integration to get

$$\int_0^1 \int_{-x}^x (x+y)^2 \, dy \, dx = \int_0^1 \left[\frac{(x+y)^3}{3} \Big|_{-x}^x \right] dx = \int_0^1 \frac{8x^3}{3} dx = \frac{2x^4}{3} \Big|_0^1 = \frac{2}{3}$$



(a)



(d)

2d. The region of integration for $\int_0^1 \int_{\tan^{-1} y}^{\pi/4} \sec^5 x \, dx \, dy$ is shown in the figure above. Changing the order of integration gives

$$\begin{aligned} \int_0^{\pi/4} \int_0^{\tan x} \sec^5 x \, dy \, dx &= \int_0^{\pi/4} \left[y \sec^5 x \Big|_0^{\tan x} \right] dx = \int_0^{\pi/4} \tan x \sec^5 x \, dx = \frac{\sec^5 x}{5} \Big|_0^{\pi/4} \\ &= \frac{1}{5} (2^{5/2} - 1) \end{aligned}$$

5. For $\iint_R \frac{dx \, dy}{x^2 + y^2 + 1}$ where R is the rectangle $[-1, 1] \times [-1, 2]$, the area of R is 6 and the function $\frac{1}{x^2 + y^2 + 1}$ takes its minimum value $m = \frac{1}{6}$ when the denominator is maximum, at $(\pm 1, 2)$, and it takes its maximum value $M = 1$ when the denominator is minimum, at $(0, 0)$. So the value of the integral lies between $m \cdot \text{Area}(R) = 1$ and $M \cdot \text{Area}(R) = 6$.

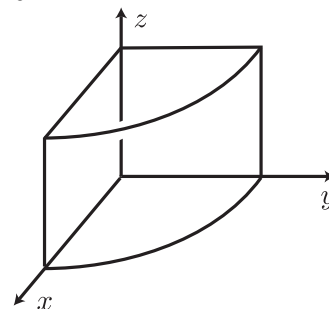
Section 5.5:

1. $\int_0^1 \int_0^1 \int_0^1 x^2 \, dx \, dy \, dz = \int_0^1 \int_0^1 \left[\frac{x^3}{3} \Big|_0^1 \right] dy \, dz = \frac{1}{3} \int_0^1 \int_0^1 dy \, dz = \frac{1}{3}$ since $\int_0^1 \int_0^1 dy \, dz$ is equal to the area of the square $[0, 1] \times [0, 1]$ in the yz -plane.

3. $\int_0^1 \int_{-1}^1 \int_0^2 (2x + 3y + z) \, dx \, dy \, dz = \int_0^1 \int_{-1}^1 \left[(x^2 + 3xy + xz) \Big|_0^2 \right] dy \, dz = \int_0^1 \int_{-1}^1 (4 + 6y + 2z) \, dy \, dz = \int_0^1 \left[(4y + 3y^2 + 2yz) \Big|_{-1}^1 \right] dz = \int_0^1 (8 + 4z) \, dz = (8z + 2z^2) \Big|_0^1 = 10$

16. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^1 z \, dz \, dy \, dx = \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\frac{z^2}{2} \Big|_0^1 \right] dy \, dx = \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} dy \, dx.$

The integral $\int_0^1 \int_0^{\sqrt{1-x^2}} dy \, dx$ is equal to the area of the shadow in the xy -plane, one-quarter of a disk of radius 1, so its area is $\frac{\pi}{4}$ and hence the final answer is $\frac{\pi}{8}$.

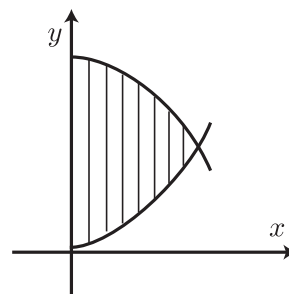


Additional Problems

A1. Sketch the region of integration and compute the integral $\iint_R 2xy \, dx \, dy$ where R is the region defined by the inequalities $x \geq 0$, $y \geq x^2$, and $y \leq 9 - x^2$.

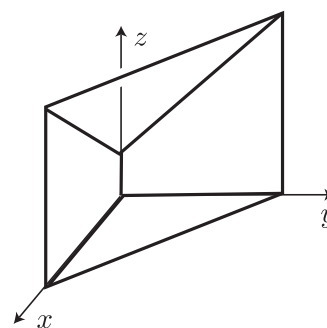
Solution. The parabolas intersect where $x^2 = 9 - x^2$, so $2x^2 = 9$ and $x = 3/\sqrt{2}$. The integral is then

$$\begin{aligned} \int_0^{3/\sqrt{2}} \int_{x^2}^{9-x^2} xy \, dy \, dx &= \int_0^{3/\sqrt{2}} \left[\frac{xy^2}{2} \Big|_{x^2}^{9-x^2} \right] dx = \int_0^{3/\sqrt{2}} \left(\frac{81x}{2} - 9x^3 \right) dx \\ &= \left(\frac{81x^2}{4} - \frac{9x^4}{4} \right) \Big|_0^{3/\sqrt{2}} = \frac{729}{16} \end{aligned}$$



A2. Sketch the region of integration and determine the limits of integration for computing $\iiint_R xyz \, dV$ over the region R bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 4$, and $z - x - y = 1$. You do not have to compute the value of the integral.

Solution. $\int_0^4 \int_0^{4-x} \int_0^{x+y+1} xyz \, dz \, dy \, dx$



A3. Let R be the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$. Write down triple integrals in the order $dz \, dx \, dy$ and $dz \, dy \, dx$ that compute the volume of R . Do not evaluate these integrals.

Solution. Where the paraboloid and the plane intersect we have $x^2 + y^2 = 2y$, which can be rewritten as $x^2 + (y-1)^2 = 1$, so the shadow of R in the xy -plane is the disk inside this circle. Solving $x^2 + (y-1)^2 = 1$ for y gives $y = 1 \pm \sqrt{1-x^2}$, so the volume of R is the triple

integral $\int_{-1}^1 \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} \int_{x^2+y^2}^{2y} dz \, dy \, dx$. Switching dy and dx , we solve $x^2 + y^2 = 2y$ for x to

get $x = \pm \sqrt{2y - y^2}$ so the volume can also be computed as $\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_{x^2+y^2}^{2y} dz \, dx \, dy$.

A4. Evaluate $\int_1^e \int_1^e \int_1^e \frac{1}{xyz} \, dx \, dy \, dz$.

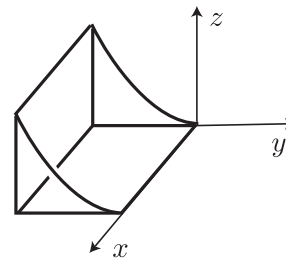
Solution. This is completely routine. The answer is 1.

A5. (a) Sketch the region of integration for the integral $\int_0^1 \int_{-1}^0 \int_0^{y^2} dz \, dy \, dx$.

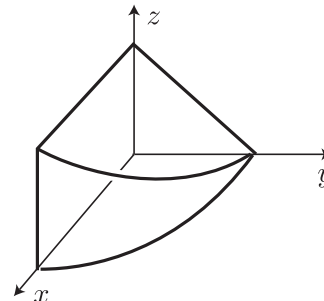
(b) Rewrite the integral as an integral in the other five orders: $dy \, dz \, dx$, $dy \, dx \, dz$, $dx \, dy \, dz$, $dx \, dz \, dy$, and $dz \, dx \, dy$

Solution. From the figure we see that the other ways to write

$$\begin{aligned} \text{the integral are } & \int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy dz dx, \int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy dx dz, \\ & \int_0^1 \int_{-1}^{-\sqrt{z}} \int_0^1 dx dy dz, \int_{-1}^0 \int_0^{y^2} \int_0^1 dx dz dy, \int_{-1}^0 \int_0^1 \int_0^{y^2} dz dx dy. \end{aligned}$$



A6. Compute $\iiint_R x dV$ where R is the region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the parabolic cylinder $x = 4 - y^2$.



Solution.

$$\begin{aligned} \int_0^2 \int_0^{4-y^2} \int_0^{2-y} x dz dx dy &= \int_0^2 \int_0^{4-y^2} [xz]_0^{2-y} dx dy = \int_0^2 \int_0^{4-y^2} (2x - xy) dx dy \\ &= \int_0^2 \left[x^2 - \frac{x^2 y}{2} \right]_0^{4-y^2} dy = \int_0^2 (16 - 8y^2 + y^4 - 8y + 4y^3 - \frac{y^5}{2}) dy = \frac{176}{15} \end{aligned}$$

A7. Evaluate the following integral by changing the order of integration in an appropriate way:

$$\int_0^4 \int_0^1 \int_{2y}^2 \frac{4 \cos(x^2)}{2\sqrt{z}} dx dy dz$$

Solution. Since $\cos(x^2)$ has no antiderivative expressible in elementary terms, we can try integrating dy before dx . This gives

$$\begin{aligned} \int_0^4 \int_0^2 \int_0^{x/2} \frac{4 \cos(x^2)}{2\sqrt{z}} dy dx dz &= \int_0^4 \int_0^2 \frac{2x \cos(x^2)}{2\sqrt{z}} dx dz = \int_0^4 \left[\frac{\sin(x^2)}{2\sqrt{z}} \right]_0^2 dz \\ &= \int_0^4 \frac{\sin 4}{2} z^{-1/2} dz = (\sin 4) z^{1/2} \Big|_0^4 = 2 \sin 4 \end{aligned}$$