Additional problems:

A1. Let $g(x,y) = x^2+2y^2$, and C be the level curve $x^2+2y^2 = 1$. Then $\nabla f = (y,x)$ and $\nabla g = (2x,4y)$. The Lagrange multiplier conditions are $y = 2\lambda x$ and $x = 4\lambda y$. Substituting the first equation into the second, we obtain $x = 4\lambda y = 4\lambda(2\lambda x) = 8\lambda^2 x$. Now, if $x = 0$, then the Lagrange multiplier conditions imply that $y = 0$, but the point $(x,y) = (0,0)$ is not on the surface C. So we can assume that $x \neq 0$. Then the equation above implies that $8\lambda^2 = 1$, so $\lambda = \pm 1/\sqrt{8}$.

Combining $y = 2\lambda x$ with the equation for C, we have

$$1 = x^2 + 2y^2 = x^2 + 2\,(2\lambda x)^2 = x^2 + 8\lambda^2 x^2 = (1+8\lambda^2)x^2,$$

so $x = \pm 1/\sqrt{1+8\lambda^2}$. Then $y = 2\lambda x = 2\lambda(\pm 1/\sqrt{1+8\lambda^2}) = \pm 2\lambda/\sqrt{1+8\lambda^2}$.

For $\lambda = 1/\sqrt{8}$, we obtain the two critical points $(x,y) = (\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$. For $\lambda = -1/\sqrt{8}$, we obtain the critical points $(x,y) = (\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$.

Now we test f at the critical points. We have $f\left(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}\right) = 1/2$ and $f\left(\pm 1/\sqrt{2}, \mp 1/\sqrt{2}\right) = -1/2$. Therefore, the maximum value $1/2$ occurs at the first two critical points, and the minimum value $-1/2$ occurs at the last two.

A2. The level curves of $f(x,y) = xy$ reveal that f increases as one moves from the origin diagonally up and to the right or down and to the left, and decreases as one moves from the origin diagonally up and to the left or down and to the right. Overlaying the unit circle onto the level curves, we can see how the value of f changes along the circle, revealing that the maximum of f occurs at $\left(1/\sqrt{2}, 1/\sqrt{2}\right)$ and $\left(-1/\sqrt{2}, -1/\sqrt{2}\right)$, and the minimum at $\left(1/\sqrt{2}, -1/\sqrt{2}\right)$ and $\left(-1/\sqrt{2}, 1/\sqrt{2}\right)$.

Let $g(x,y) = x^2+y^2$, and C be the level curve $x^2+y^2 = 1$. Then $\nabla f = (y,x)$ and $\nabla g = (2x,2y)$. The Lagrange multiplier conditions are $y = 2\lambda x$ and $x = 2\lambda y$. Substituting the first equation into the second, we obtain $x = 2\lambda y = 2\lambda(2\lambda x) = 4\lambda^2 x$.

For the same reason as in the previous problem, we can assume that $x \neq 0$, so $1 = 4\lambda^2$, so $\lambda = \pm 1/2$.

Combining $y = 2\lambda x$ with the equation for C, we have

$$1 = x^2 + y^2 = x^2 + (2\lambda x)^2 = x^2 + 4\lambda^2 x^2 = (1 + 4\lambda^2)x^2,$$

so $x = \pm 1/\sqrt{1 + 4\lambda^2}$. Then $y = 2\lambda x = 2\lambda(\pm 1/\sqrt{1 + 4\lambda^2}) = \pm 2\lambda/\sqrt{1 + 4\lambda^2}$.

For $\lambda = 1/2$, we obtain the two critical points $(x, y) = (\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$. For $\lambda = -1/\sqrt{2}$, we obtain the critical points $(x, y) = (\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$.

Now we test f at the critical points. As in the previous problem, we have $f\left(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}\right) = 1/2$ and $f\left(\pm 1/\sqrt{2}, \mp 1/\sqrt{2}\right) = -1/2$. Therefore, the maximum value $1/2$ occurs at the first two critical points, and the minimum value $-1/2$ occurs at the last two.

Problems from the book:

Section 3.4:

11. First we consider the boundary of the unit disk. Let $g(x, y) = x^2 + y^2$, and C be the level curve $x^2 + y^2 = 1$. Then $\nabla f = (2x + y, x + 2y)$ and $\nabla g = (2x, 2y)$. The Lagrange multiplier conditions are $2x + y = 2\lambda x$ and $x + 2y = 2\lambda y$. The first equation tells us that $y = 2\lambda x - 2x$, and substituting this into the second equation yields $x + 2(2\lambda x - 2x) = 2\lambda(2\lambda x - 2x)$, or $x(2\lambda - 3)(2\lambda - 1) = 0$. So $x = 0$ or $\lambda = 3/2$ or $\lambda = 1/2$. But if we plug $x = 0$ into the first Lagrange multiplier condition, we see that $y = 0$ also. Since $(x, y) = (0, 0)$ is not on the curve C, we can assume that $x \neq 0$. Substituting $y = 2\lambda x - 2x$ into the equation $x^2 + y^2 = 1$ tells us that

$$1 = x^2 + (2\lambda x - 2x)^2 = x^2 + 4x^2(\lambda - 1)^2 = x^2\left(1 + 4(\lambda - 1)^2\right),$$

so $x = \pm 1/\sqrt{1 + r(\lambda - 1)^2}$.

For $\lambda = 3/2$, we obtain $x = \pm 1/\sqrt{2}$ and $y = 2\lambda x - 2x = 2(3/2)(\pm 1/\sqrt{2}) - 2(\pm 1/\sqrt{2}) = \pm 1/\sqrt{2}$. For $\lambda = 1/2$, we again obtain $x = \pm 1/\sqrt{2}$, but this time $y = 2\lambda x - 2x = 2(1/2)(\pm 1/\sqrt{2}) - 2(\pm 1/\sqrt{2}) = \mp 1/\sqrt{2}$. So we have four critical points for $f|_C$,

$$\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right), \quad \left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right).$$

Testing f at these points, we have $f\left(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}\right) = 3/2$ and $f\left(\pm 1/\sqrt{2}, \mp 1/\sqrt{2}\right) = 1/2$. Therefore, the maximum value of f on C is $3/2$, occurring at the first two critical points, and the minimum value is $1/2$ occurring at the last two.

Now we consider the interior of the disk. The gradient of f is zero when $2x + y = 0$ and $x + 2y = 0$, and the only solution to this set of equations is $(x, y) = (0, 0)$, so this is the only critical point in the interior of the disk, and the function value here is $f(0, 0) = 0$.

Comparing the values of f at all of the critical points we have found, we see that minimum value of f on the closed unit disk is 0, attained at $(0, 0)$, and the maximum vale is $3/2$, attained at the two points $\left(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}\right)$.

12. Let x, y, and z be the length, width, and height of the box, respectively. Then the volume of the box is $V(x, y, z) = xyz$ and the surface area is $A(x, y, z) = xy + 2xz + 2yz$ (remember that the box has no top). Denote the level surface $A(x, y, z) = 16$ by S. If any of x, y, z actually equals zero, then $V = 0$. Since we can assume that the maximum possible volume is actually greater than zero, we can assume throughout that $x, y, z \neq 0$.

We have $\nabla V = (yz, xz, xy)$ and $\nabla A = (y + 2x, x + 2z, 2x + 2y)$, so the Lagrange multiplier conditions are $yz = \lambda(y + 2z)$, $xz = \lambda(x + 2z)$, and $xy = \lambda(2x + 2y)$. Solving the first equation for λ yields $\lambda = yz/(y + 2z)$, and substituting this into the second equation gives us $xz = (yz/(y + 2z))\,(x + 2z)$. Canceling a z from both sides and cross-multiplying gives $x(y + 2z) = y(x + 2z)$, or $xy + 2xz = xy + 2yz$. Subtracting xy and dividing by $2z$ on both sides, we see that $x = y$. Substituting our expression for λ into the third equation, we similarly conclude that $x = 2z$, of $z = x/2$. Plugging this information into the equation for S, we have

$$16 = xy + 2xz + 2yz = x(x) + 2x(x/2) + 2x(x/2) = 3x^2,$$

so $x = \pm 4/\sqrt{3}$. Since we are only considering positive values of x, we conclude that $x = 4/\sqrt{3}$, and hence $y = 4/\sqrt{3}$ and $z = 2/\sqrt{3}$. This is the only critical point, so this must be where V takes its maximum. (Technically, x, y, and z are measures of length in meters.)

15. Note first that $\nabla f = (1, 1, -1)$, so f has no critical points, so in particular it has no critical points on the inside of the unit sphere.

Let $g(x, y, z) = x^2 + y^2 + z^2$, and denote the level curve $g(x, y, z) = 1$ by S. We calculate $\nabla g = (2x, 2y, 2z)$, so the Largrangian multiplier conditions are $1 = 2\lambda x$, $1 = 2\lambda y$, and $-1 = 2\lambda z$. Notice that λ cannot equal zero, so we conclude that $x = 1/(2\lambda)$, $y = 1/(2\lambda)$, and $z = -1/(2\lambda)$. Plugging this into the equation for the level curve S, we have

$$1 = x^2 + y^2 + z^2 = \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 = \frac{3}{4\lambda^2},$$

so $\lambda^2 = 3/4$ and $\lambda = \pm\sqrt{3}/2$. This leads us to the two critical points $\left(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \mp 1/\sqrt{3}\right)$. We calculate

$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}} = \sqrt{3},$$

$$f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = -\frac{3}{\sqrt{3}} = -\sqrt{3},$$

so the maximum of sphere on the closed unit ball is $\sqrt{3}$, attained at the first critical point, and its minimum is $-\sqrt{3}$, attained at the second.

17. Let x be the length of the vertical sides and y be the length of the horizontal sides. The cost of the trim is given by $C = 2qx + 2py$. We are given that $A = xy$, so

$y = A/x$, so

$$C = 2qx + 2py = 2qx + 2p\left(\frac{A}{x}\right) = 2qx + \frac{2pA}{x}.$$

So C is just a function of the single variable x, and we are trying to minimize it on the domain $0 < x < \infty$. We have $A'(x) = 2q - 2pA/x^2$, so the only critical point occurs when $2q = 2pA/x^2$, which is when $x^2 = pA/q$, which is when $x = \pm\sqrt{pA/q}$. On our domain, the only critical point is $x = \sqrt{pA/q}$. It is easy to check that $A'(x)$ is negative when $0 < x < \sqrt{pA/q}$ and positive when $x > \sqrt{pA/q}$, so $A(x)$ changes from decreasing to increasing at our critical point, so it must be a minimum. So the total cost is minimized when $x = \sqrt{pA/q}$ inches and $y = A/x = A/\sqrt{pA/q} = \sqrt{qA/p}$ inches.

This problem could also be done with Lagrangian multipliers, of course.

21. Let x, y, and z be the length, width, and height of the box, respectively, let $g(x, y, z) = x + 2y + 2z$, and denote the level surface $g(x, y, z) = 108$ by S. We are trying to maximize the function f subject to the constraints $g(x, y, z) \leq 108$ and $x, y, z \geq 0$. If any of x, y, z actually equals zero, then $f = 0$. Since we can assume that the maximum possible volume is actually greater than zero, we can assume throughout that $x, y, z \neq 0$.

We have $\nabla f = (yz, xz, xy)$. Since we are assuming that $x, y, z \neq 0$, we see that f has no critical points in the domain we are considering. So now we look for critical points of $f|_S$.

We calculate $\nabla g = (1, 2, 2)$, so the Lagrangian multiplier conditions are $yz = \lambda$, $xz = 2\lambda$, and $xy = 2\lambda$. Combining the first two equations, we have $xz = 2\lambda = 2yz$, so $x = 2y$. Combining the last two equations, we have $xz = xy$, so $z = y$. Plugging these into the equation for S, we have $108 = x + 2y + 2z = 2y + 2y + 2y = 6y$, so $y = 108/6 = 18$. Hence $x = 2(18) = 36$ and $z = 18$. This is the only critical point of $f|_S$, so it must be here that f takes its maximum on the specified domain. The volume of this maximum-volumed-box is $f(36, 18, 18) = 36(18)(18) = 11664$

inches.

22. Let us write $P = (x_0, y_0, z_0)$. The function that is being maximized is $(x, y, z) \mapsto \sqrt{x^2 + y^2 + z^2}$, and P is the point where this function is maximized when restricted to the surface S. But this is equivalent to maximizing the function $d(x, y, z) = x^2 + y^2 + z^2$, the distance squared from the origin. By Theorem 9, page 227, we know that $\nabla d(P)$ is perpendicular to S at P. But $\nabla d = (2x, 2y, 2z)$, so $\nabla d(P) = (2x_0, 2y_0, 2z_0) = 2(x_0, y_0, z_0)$, so P is parallel to $\nabla d(P)$, and hence perpendicular to S at P.

27. Let $f(x, y, z) = xy + yz$ and $g(x, y, z) = xz$, and denote the level surface $xz = 1$ by S. We have $\nabla f = (y, x + z, y)$ and $\nabla g = (z, 0, x)$, so the Lagrangian multiplier conditions are $y = \lambda z$, $x + z = \lambda(0) = 0$, and $y = \lambda x$. The middle equation implies that $z = -x$, and plugging this into the equations for S, we have $1 = xz = x(-x) = -x^2$, so $x^2 = -1$. This equation has no real solutions, so $f|_S$ has no critical points, so f has neither maximum nor minimum when restricted to S.

Section 3.5:

2. Let $F(x, y, z) = xy + z + 3xz^5 - 4$, and note that $F(1, 0, 1) = 0$. We have $\frac{\partial F}{\partial z} = 1 + 15xz^4$, so $\frac{\partial F}{\partial z}(1, 0, 1) = 1 + 15 = 16$. Then by the Special Implicit Function Theorem, for (x, y) near $(1, 0)$ and z near 1, we can write z as a function $z = g(x, y)$ of x and y. We calculate

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial g}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{y + 3z^5}{1 + 15xz^4}, \\ \frac{\partial z}{\partial y} &= \frac{\partial g}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} = -\frac{x}{1 + 15xz^4}. \end{aligned}$$

So at $(x, y, z) = (1, 0, 1)$, we have $\frac{\partial z}{\partial x} = (0+3)/(1+15) = 3/16$ and $\frac{\partial z}{\partial y} = 1/(1+15) = 1/16$.

3.

(a) The equation $y^2 + y + 3x + 1 = 0$ can be solved for y by using the quadratic formula, where $A = 1$, $B = 1$, and $C = 3x + 1$. We have

$$y = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-1 \pm \sqrt{1 - 4(3x+1)}}{2} = \frac{-1 \pm \sqrt{-12x - 3}}{2}.$$

This formula yields a real number solution exactly when $-12x - 3 \geq 0$, which is when $-12x \geq 3$, which is when $x \leq -1/4$.

(b) We compute $\frac{\partial F}{\partial y} = 2y + 1$, so by the Special Implicit Function Theorem we can write y as a function of x if $2y + 1 \neq 0$, which is when $y \neq -1/2$.

The main equation can be rewritten as $y^2 + y + 1 = -3x$. Analyzing $y^2 + y + 1$ as an expression for a parabola, we see that it has its smallest value when $y = -1/2$, and all other values are strictly larger. The value of the expression at $y = -1/2$ is $(-1/2)^2 - (1/2) + 1 = 3/4$. Therefore, if $y \neq -1/2$, then $-3x = y^2 + y + 1 > 3/4$, so $x < -1/4$. This agrees exactly with our result from Part (a).

7. Let $F_1(x, y, u, v) = y + x + uv$, $F_2(x, y, u, v) = uxy + v$, and $F(x, y, u, v) = (F_1(x, y, u, v), F_2(x, y, u, v))$. Then

$$\begin{bmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{bmatrix} = \begin{bmatrix} v & u \\ xy & 1 \end{bmatrix},$$

and its determinant is $v - uxy$. At $(x, y, u, v) = (0, 0, 0, 0)$, the determinant is 0. Hence the Implicit Function Theorem does not apply in this case.

From the equation $F_2 = 0$, we have $v = -uxy$, and plugging this into the equation $F_1 = 0$ yields $0 = y + x + uv = y + x + u(-uxy) = x + y - xyu^2$. So $xyu^2 = x + y$. For (x, y) near $(0, 0)$, this solution will either have two, one, or no solutions for u. Each solutions for u can be plugged into $v = -uxy$ to obtain v.

Notice that $F(2,2,1,-4) = (0,0)$, and at this point the determinant above is $v - uxy = (-4) - (1)(2)(2) = -8$. Hence, by the Implicit Function Theorem, there is a function $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that for (x,y) near $(2,2)$ and (u,v) near $(1,-4)$ we have $(u,v) = g(x,y)$. For (x,y) near enough to $(2,2)$, $x+y$ will be nonzero and xy will be positive, so the equation $xyu^2 = x+y$ will have exactly two solutions for u, $u = \pm\sqrt{(x+y)/(xy)}$, which lead to two solutions for v, $v = \mp xy\sqrt{(x+y)/(xy)} = \mp\sqrt{(xy)(x+y)}$. But if (u,v) are sufficiently close to $(1,-4)$, then u will have to be positive and v will have to be negative, so we can only use the solutions $u = \sqrt{(x+y)/(xy)}$ and $v = -\sqrt{(xy)(x+y)}$.

8. Let $F(x,y,z) = (u(x,y,z),\, v(x,y,z),\, w(x,y,z))$. We are seeking to determine whether or not the function F can be inverted near $(x,y,z) = (0,0,0)$. This is a job for the Inverse Function Theorem. We have

$$\mathbf{D}F(x,y,z) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} 1+yz & xz & xy \\ y & x+1 & 0 \\ 2 & 0 & 6z+1 \end{bmatrix},$$

so

$$\mathbf{D}F(0,0,0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

The determinant of this last matrix is 1, so the Inverse Function Theorem tells us that near $(x,y,z) = (0,0,0)$ we can find an inverse for F, which means exactly that we can write x, y, and z as functions of u, v, and w.