Problems from the book:

Section 3.1:

3. If $f(x,y) = \cos(xy^2)$ then we have $\frac{\partial f}{\partial x} = -y^2\sin(xy^2)$ and $\frac{\partial f}{\partial y} = -2xy\sin(xy^2)$. Computing partials with respect to x for these two functions gives $\frac{\partial^2 f}{\partial x^2} = -y^4\cos(xy^2)$ and $\frac{\partial^2 f}{\partial x \partial y} = -2xy^3\cos(xy^2) - 2y\sin(xy^2)$. Computing partials with respect to y gives $\frac{\partial^2 f}{\partial y \partial x} = -2xy^3\cos(xy^2) - 2y\sin(xy^2)$ and $\frac{\partial^2 f}{\partial y^2} = -4x^2y^2\cos(xy^2) - 2x\sin(xy^2)$. In particular this shows $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

16. We are given $w = f(x,y)$ with $x = u + v$ and $y = u - v$. To compute $\frac{\partial^2 w}{\partial u \partial v}$ we will first compute $\frac{\partial w}{\partial v}$ and then $\frac{\partial}{\partial u}(\frac{\partial w}{\partial v})$. Using the chain rule, we first have

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} \quad \text{which equals} \quad \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} \quad \text{since} \quad \frac{\partial x}{\partial v} = 1 \;\; \text{and} \;\; \frac{\partial y}{\partial v} = -1.$$

For the next step of computing $\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial}{\partial u}(\frac{\partial w}{\partial v})$ we do the same procedure for each of the two functions $g = \frac{\partial w}{\partial x}$ and $h = \frac{\partial w}{\partial y}$:

$$\begin{aligned}
\frac{\partial}{\partial u}\Big(\frac{\partial w}{\partial v}\Big) &= \frac{\partial}{\partial u}\Big(\frac{\partial w}{\partial x}\Big) - \frac{\partial}{\partial u}\Big(\frac{\partial w}{\partial y}\Big) \\
&= \bigg[\frac{\partial}{\partial x}\Big(\frac{\partial w}{\partial x}\Big)\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\Big(\frac{\partial w}{\partial x}\Big)\frac{\partial y}{\partial u}\bigg] - \bigg[\frac{\partial}{\partial x}\Big(\frac{\partial w}{\partial y}\Big)\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\Big(\frac{\partial w}{\partial y}\Big)\frac{\partial y}{\partial u}\bigg] \\
&= \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y \partial x} - \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2} \quad \text{since} \quad \frac{\partial x}{\partial u} = 1 \;\; \text{and} \;\; \frac{\partial y}{\partial u} = 1 \\
&= \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}
\end{aligned}$$

17. We are given two functions $f(x,y)$ and $c(t) = (x(y), y(t))$ which can be composed to form a function of one variable $f(c(t))$ whose first derivative is $\frac{d}{dt}\big[f(c(t))\big] = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ and whose second derivative can be computed from the first derivative using the product rule first and then the chain rule:

$$\begin{aligned}
\frac{d^2}{dt^2}\big[f(c(t))\big] &= \frac{d}{dt}\Big(\frac{\partial f}{\partial x}\Big)\frac{dx}{dt} + \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \frac{d}{dt}\Big(\frac{\partial f}{\partial y}\Big)\frac{dy}{dt} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2} \\
&= \Big(\frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dt}\Big)\frac{dx}{dt} + \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \Big(\frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}\Big)\frac{dy}{dt} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2} \\
&= \frac{\partial^2 f}{\partial x^2}\Big(\frac{dx}{dt}\Big)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt}\frac{dy}{dt} + \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial^2 f}{\partial y^2}\Big(\frac{dy}{dt}\Big)^2 + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2}
\end{aligned}$$

Section 3.2:

2. $f(x,y) = (x^2+y^2+1)^{-1}$ at $(x_0,y_0)=(0,0)$. We need to compute the first and second partials at $(0,0)$. We have $\frac{\partial f}{\partial x} = -(x^2+y^2+1)^{-2}(2x)$, so $\frac{\partial f}{\partial x}(0,0)=0$. Since f is symmetric in x and y this implies that $\frac{\partial f}{\partial y}(0,0)=0$ as well (or you can just do the calculation). For the second partials we have

$$\begin{aligned} &\frac{\partial^2 f}{\partial x^2} = -(x^2+y^2)^{-2}(2) - (2x)(-2)(x^2+y^2)^{-3}(2x) \quad \text{so} \quad \frac{\partial^2 f}{\partial x^2}(0,0) = -2 \\ &\frac{\partial^2 f}{\partial x \partial y} = 2(x^2+y^2)^{-3}(2x)(2y) \quad \text{so} \quad \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0 \\ &\text{and} \quad \frac{\partial^2 f}{\partial y^2}(0,0) = 0 \quad \text{by symmetry.} \end{aligned}$$

The second order part of the Taylor series at $(0,0)$ is then

$$\begin{aligned} &f(0,0) + f_x(0,0)\,x + f_y(0,0)\,y + \frac{1}{2}f_{xx}(0,0)\,x^2 + f_{xy}(0,0)\,xy + \frac{1}{2}f_{yy}(0,0)\,y^2 \\ &= 1 - x^2 - y^2 \end{aligned}$$

With the book's terminology, the remainder term R_2 would be tacked on at the end here.

3. If $f(x,y) = e^{x+y}$ then all the various partial derivatives of f (of any order) are also equal to e^{x+y}, so evaluating at $(0,0)$ always gives 1. The Taylor series then begins

$$1 + x + y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + \cdots$$

Note that this could also be obtained from the one-variable Taylor series (Maclaurin series) for e^x by simply replacing x by $x+y$.

Section 3.3:

1. $f(x,y) = x^2 - y^2 + xy$. To find critical points we set both first partials equal to zero: $f_x = 2x + y = 0$ and $f_y = -2y + x = 0$. These are the equations for two lines through the origin having different slopes, so the only common solution is the origin, $(x,y)=(0,0)$. The Hessian matrix is

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

This has negative determinant, -5, so the critical point $(0,0)$ is a saddle.

2. $f(x,y) = x^2 + y^2 - xy$. The critical points satisfy $f_x = 2x - y = 0$ and $f_y = 2y - x = 0$. Again these are two distinct lines through the origin, so the only solution is $(0,0)$. The Hessian matrix is

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

The determinant is 3 which is positive so the critical point $(0,0)$ is either a local min or a local max. Since $f_{xx} > 0$ it must be a local minimum.

7. $f(x,y) = 3x^2 + 2xy + 2x + y^2 + y + 4$. The critical points satisfy $f_x = 6x + 2y + 2 = 0$ and $f_y = 2x + 2y + 1 = 0$. These two equations have the solution $(x,y) = (-\frac{1}{4}, -\frac{1}{4})$. The Hessian is

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 2 \end{bmatrix}$$

with positive determinant 8, and $f_{xx} > 0$ so the critical point $(-\frac{1}{4}, -\frac{1}{4})$ is a local minimum.

8. $f(x,y) = \sin(x^2 + y^2)$. At critical points we must have $f_x = 2x\cos(x^2 + y^2) = 0$ and $f_y = 2y\cos(x^2 + y^2) = 0$. The circles $x^2 + y^2 = \pi/2, 3\pi/2, 5\pi/2, \cdots$ therefore consist of critical points, and if (x,y) is not on any of these circles then $\cos(x^2 + y^2) \neq 0$ and we must have $x = 0$ and $y = 0$ so $(0,0)$ is the only other critical point. For the critical point $(0,0)$ we apply the second derivative test:

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} -4x^2\sin(x^2+y^2) + 2\cos(x^2+y^2) & -4xy\sin(x^2+y^2) \\ -4xy\sin(x^2+y^2) & -4y^2\sin(x^2+y^2) + 2\cos(x^2+y^2) \end{bmatrix}$$

which equals $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ at $(0,0)$. This has positive determinant and $f_{xx} = 2 > 0$ so $(0,0)$ is a local minimum. On the circles $x^2 + y^2 = \pi/2, 3\pi/2, 5\pi/2, \cdots$ the value of $f(x,y)$ is alternately 1 and -1. These are the maximum and minimum values of $f(x,y)$ overall, so the points of these circles are all local maxima or local minima, depending on whether $f = 1$ or $f = -1$ on the circle.

11. $f(x,y) = e^x \cos y$. At a critical point we have $f_x = e^x \cos y = 0$ and $f_y = -e^x \sin y = 0$. Since e^x is never 0 we must have $\sin y = 0$ and $\cos y = 0$, but there are no values of y for which both $\sin y$ and $\cos y$ are zero. (Just look at the graphs of sin and cos.) Thus there are no critical points.

19. (a) For $f(x,y) = (y - 3x^2)(y - x^2) = y^2 - 4x^2y + 3x^4$ we have $f_x = 12x^3 - 8xy$ and $f_y = 2y - 4x^2$. Both partials are 0 at $(x,y) = (0,0)$, so $(0,0)$ is a critical point of f.

(b) If we let $g(t) = (at, bt)$ then the function $h(t) = f(g(t)) = (bt - 3a^2t^2)(bt - a^2t^2) = b^2t^2 - 4ba^2t^3 + 3a^4t^4$ has a critical point at $t = 0$ since its first derivative is $h'(t) = 2b^2t - 12ba^2t^2 + 12a^4t^3$ which is 0 at $t = 0$. The second derivative is $h''(t) = 2b^2 - 24ba^2t + 36a^4t^2$, so $h''(0) = 2b^2 > 0$ if $b \neq 0$, so the critical point of h at 0 is a local minimum if $b \neq 0$. If $b = 0$ the function h reduces to $h(t) = 3a^4t^4$ which also has a local minimum at 0.

(c) In spite of what we have just shown in part (b), the critical point of f at $(0,0)$ is actually a saddle. To see that this is true, note that $f(0,0)=0$ so what we have to show is that f takes both positive and negative values arbitrarily close to $(0,0)$. Certainly f takes on positive values arbitrarily close to $(0,0)$ since along the y axis we have $f(0,0)=y^2$, for example. To see that f also takes on negative values near $(0,0)$, note that the product $(y-3x^2)(y-x^2)$ will be negative if the two factors have opposite sign, and this will happen if the value of y lies between x^2 and $3x^2$, so we have $x^2 < y < 3x^2$. This means that f is negative in the region between the two parabolas $y=x^2$ and $y=3x^2$. This region contains points arbitrarily close to $(0,0)$ so we are done.

22. The point on the plane $2x-y+2z=20$ closest to the origin is a minimum of the function $x^2+y^2+z^2$. Thus we have a constrained extremum problem, minimizing $x^2+y^2+z^2$ with the constraint $2x-y+2z=20$, which we would usually solve using the Lagrange multiplier method, the topic of the following section, but we can avoid this by first solving $2x-y+2z=20$ for one of the variables in terms of the other two and then substituting this into $x^2+y^2+z^2$ to get a 2-variable function to minimize. It is easiest to solve $2x-y+2z=20$ for y, obtaining $y=2x+2z-20$. Substituting into $x^2+y^2+z^2$, we get the function $f(x,z)=x^2+(2x+2z-20)^2+z^2$. To find critical points of f we compute $f_x=10x+8z-80$ and $f_z=8x+10z-80$. Setting both these quantities equal to 0 and solving, we find $x=z=40/9$. The equation $y=2x+2z-20$ then gives $y=-20/9$. Thus the function f has only one critical point, $(40/9,-20/9,40/9)$. This has to give the closest point to the origin on the plane since it is obvious from geometry that a closest point exists, so f does have a minimum.

23. Let the dimensions of the box be x, y, and z, so its volume is $V=xyz$, a given constant, and its area is $2xy+2xz+2yz$, a function we want to minimize with the constraint $xyz=V$. As in the previous problem we use the constraint equation to reduce to an unconstrained 2-variable problem. Substituting $z=\frac{V}{xy}$ into the area formula gives the function $f(x,y)=2xy+2Vy^{-1}+2Vx^{-1}$ that we wish to minimize. We have $f_x=2y-2Vx^{-2}=0$ and $f_y=2x-2Vy^{-2}=0$. Thus $y=Vx^{-2}$ and $x=Vy^{-2}$. Subtituting the first of these two equations into the second gives $x=V(x^4/V^2)=x^4/V$. Since $x\neq 0$ (the box can't have volume 0), we can divide by x to get $1=x^3/V$, so $x=V^{1/3}$. Then $y=Vx^{-2}$ gives $y=V^{1/3}$, and the original equation $xyz=V$ then gives $z=V^{1/3}$. Thus we have $x=y=z$ and the box is a cube.

(We are assuming here that the area of the box *does* have a minimum value. We could use the second derivative test to show the critical point we found is a local minimum, but that wouldn't show it is a global minimum. To show there is a global minimum we would have to show that the area approaches infinity if any of the coordinates x, y, z approach infinity. A similar question is discussed on pp. 231-232 of the book.)

Additional problems:

A1. Verify equality of mixed partials $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ first for the function $f(x,y) = x^m y^n$ and then for an arbitrary function $f(x,y) = g(x)h(y)$.

Solution. For $f(x,y) = x^m y^n$ we have $\frac{\partial f}{\partial x} = mx^{m-1}y^n$, $\frac{\partial^2 f}{\partial y \partial x} = mnx^{m-1}y^{n-1}$, $\frac{\partial f}{\partial y} = nx^m y^{n-1}$, and $\frac{\partial^2 f}{\partial x \partial y} = mnx^{m-1}y^{n-1}$, so $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. More generally, if $f(x,y) = g(x)h(y)$ then $\frac{\partial f}{\partial x} = g'(x)h(y)$, $\frac{\partial^2 f}{\partial y \partial x} = g'(x)h'(y)$, $\frac{\partial f}{\partial y} = g(x)h'(y)$, and $\frac{\partial^2 f}{\partial x \partial y} = g'(x)h'(y)$, so again $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

A2. Find all local minima, maxima, and saddles for the function $f(x,y) = x^3 - y^3 - 2xy$.

Solution. First find the critical points: $f_x = 3x^2 - 2y = 0$, $f_y = -3y^2 - 2x = 0$. The first equation gives $y = 3x^2/2$ and plugging this into the second equation leads to $27x^4 + 8x = 0$. This has solutions $x = 0$ and $x = -2/3$. The corresponding y values are $y = 0$ and $y = 2/3$. Thus we have two critical points $(0,0)$ and $(-2/3, 2/3)$. To decide whether they are local maxima, minima or saddles we compute the Hessian matrix:

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & -2 \\ -2 & -6y \end{bmatrix}$$

At $(0,0)$ this is $\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$ which has negative determinant so this critical point is a saddle. At $(-2/3, 2/3)$ the Hessian matrix is $\begin{bmatrix} -4 & -2 \\ -2 & -4 \end{bmatrix}$ which has positive determinant, so this critical point is a local maximum since $f_{xx} = -4 < 0$.