<u>Problems from the book, section 2.3:</u>

1. (a) If $f(x,y) = xy$ then $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = x$.
(c) If $f(x,y) = x\cos x\cos y$ then using the product rule for derivatives we have $\frac{\partial f}{\partial x} = x(-\sin x)\cos y + 1\cos x\cos y = -x\sin x\cos y + \cos x\cos y$. For $\frac{\partial f}{\partial y}$ we don't need the product rule, and we get $\frac{\partial f}{\partial y} = x\cos x(-\sin y) = -x\cos x\sin y$.

2. (a) For $z = (a^2 - x^2 - y^2)^{1/2}$ we have

$$\frac{\partial z}{\partial x} = \frac{1}{2}(a^2 - x^2 - y^2)^{-1/2}(-2x) = -x/\sqrt{a^2 - x^2 - y^2}$$

Plugging in the values $(x,y) = (0,0)$ and $(x,y) = (a/2, a/2)$ gives $\frac{\partial z}{\partial x}(0,0) = 0$ and $\frac{\partial z}{\partial x}(a/2,a/2) = -(a/2)/\sqrt{a^2/2} = -\sqrt{2}/2$. Since everything in this problem is symmetric when x and y are interchanged, we also get $\frac{\partial z}{\partial y}(0,0) = 0$ and $\frac{\partial z}{\partial y}(a/2,a/2) = -\sqrt{2}/2$.

3. (a) If $w = xe^{x^2+y^2}$ then using the product rule, $\frac{\partial w}{\partial x} = xe^{x^2+y^2}(2x) + (1)e^{x^2+y^2} = (2x^2+1)e^{x^2+y^2}$. For $\frac{\partial w}{\partial y}$ we just get $\frac{\partial w}{\partial y} = 2xye^{x^2+y^2}$.

5. We know that the general equation for the tangent plane to $z = f(x,y)$ at (x_0, y_0) is $z = \frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0) + f(x_0,y_0)$. For $f(x,y) = x^2 + y^3$ we have $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 3y^2$. Plugging in the values $(x_0, y_0, f(x_0,y_0)) = (3,1,10)$ we get the plane $z = 6(x-3) + 3(y-1) + 10 = 6x + 3y - 11$.

6. (a) In problem 1(a) we had $f(x,y) = xy$, $\frac{\partial f}{\partial x} = y$, and $\frac{\partial f}{\partial y} = x$. At the point $(x_0, y_0, f(x_0,y_0)) = (0,0,0)$ we plug these numbers into the formula for the tangent plane and the result is simply $z = 0$, so the tangent plane is the xy-plane.
(c) In 1(c) we had $f(x,y) = x\cos x\cos y$, $\frac{\partial f}{\partial x} = -x\sin x\cos y + \cos x\cos y$, and $\frac{\partial f}{\partial y} = -x\cos x\sin y$. At $(x_0,y_0) = (0,\pi)$ we have $f(0,\pi) = 0$, $\frac{\partial f}{\partial x}(0,\pi) = -1$, $\frac{\partial f}{\partial y}(0,\pi) = 0$ so the tangent plane is $z = (-1)(x-0) + 0(y-\pi) + 0 = -x$.

7. (c) For $f(x,y,z) = (x + e^z + y, yx^2) = (u,v)$ the matrix of partial derivatives is

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 1 & e^z \\ 2xy & x^2 & 0 \end{bmatrix}$$

9. First we compute the tangent plane to $z = e^{x-y}$ at $(1,1,1)$. We have $\frac{\partial z}{\partial x} = e^{x-y}$ and $\frac{\partial z}{\partial y} = -e^{x-y}$. Evaluating at $x = y = 1$, these partial derivatives take the values 1 and -1. The tangent plane is then $z = 1(x-1) - 1(y-1) + 1 = x - y + 1$. To determine where this plane meets the z-axis, note that the z-axis consists of all points whose x and y coordinates are both zero. So all we have to do is set $x = 0$ and $y = 0$ in the equation for the tangent plane $z = x - y + 1$ to get the value $z = 1$. Thus the tangent plane meets the z-axis at the point $(0, 0, 1)$.

10. Two graphs that pass through the same point will be tangent if they have the same tangent plane at that point. The graphs of $f(x, y) = x^2 + y^2$ and $g(x, y) = -x^2 - y^2 + xy^3$ both pass through the origin $(0, 0, 0)$, and we can check that they have the same tangent plane at the origin by computing their partials:

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial x}(0,0) = 0, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial y}(0,0) = 0$$
$$\frac{\partial g}{\partial x} = -2x + y^3, \quad \frac{\partial g}{\partial x}(0,0) = 0, \quad \frac{\partial g}{\partial y} = -2y + 3xy^2, \quad \frac{\partial g}{\partial y}(0,0) = 0$$

Since f and g have the same partials at $(0, 0)$, they have the same tangent planes at the origin, so they are tangent.

12. (c) To approximate $\sqrt{(4.01)^2 + (3.98)^2 + (2.02)^2}$ we will first compute the linear approximation to the function $f(x, y, z) = (x^2+y^2+z^2)^{1/2}$ at the point $(x_0, y_0, z_0) = (4, 4, 2)$. We have $\frac{\partial f}{\partial x} = x/(x^2 + y^2 + z^2)^{1/2}$, so $\frac{\partial f}{\partial x}(4, 4, 2) = 2/3$. Also, $\frac{\partial f}{\partial y} = y/(x^2 + y^2 + z^2)^{1/2}$ so $\frac{\partial f}{\partial y}(4, 4, 2) = 2/3$, and similarly $\frac{\partial f}{\partial z} = z/(x^2 + y^2 + z^2)^{1/2}$ so $\frac{\partial f}{\partial z}(4, 4, 2) = 1/3$. We also have $f(4, 4, 2) = 6$. The linear approximation is then the function $\frac{2}{3}(x - 4) + \frac{2}{3}(y - 4) + \frac{1}{3}(z - 2) + 6$. We plug $(x, y, z) = (4.01, 3.98, 2.02)$ into this formula and get the approximation $\frac{2}{3}(.01) + \frac{2}{3}(-.02) + \frac{1}{3}(.02) + 6 = 6$. A calculator gives a value of 6.00007499953 which is presumably more accurate.

13. (b) We are given $f(x, y, z) = xyz/(x^2 + y^2 + z^2)$. By the quotient rule,

$$\frac{\partial f}{\partial x} = \frac{yz(x^2 + y^2 + z^2) - xyz(2x)}{(x^2 + y^2 + z^2)^2} = \frac{-x^2yz + y^3z + yz^3}{(x^2 + y^2 + z^2)^2}$$

By symmetry there are similar formulas for $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$, so

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{-x^2yz + y^3z + yz^3}{(x^2 + y^2 + z^2)^2}, \frac{x^3z - xy^2z + xz^3}{(x^2 + y^2 + z^2)^2}, \frac{x^3y + xy^3 - xyz^2}{(x^2 + y^2 + z^2)^2}\right)$$

16. If $h(x, y, z) = (x + z)e^{x-y}$ then

$$\nabla h = \big((x + z + 1)e^{x-y}, -(x + z)e^{x-y}, e^{x-y}\big)$$

and hence $\nabla h(1, 1, 1) = (3, -2, 1)$.

Additional problems:

A1. Consider the function

$$f(x,y) = \frac{x^2y^2}{x^4+y^4}, \quad \text{defined for } (x,y) \neq (0,0)$$

Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist by showing that $f(x,y)$ has different limits as (x,y) approaches $(0,0)$ along the coordinate axes and along lines $y = kx$ for various constants k.

Solution. Let's compute the values of $f(x,y)$ along the coordinate axes and along the lines $y = kx$:

$$\begin{aligned}
x\text{-axis:} \quad & f(x,0) = \frac{x^2 \cdot 0}{x^4 + 0^4} = 0 \quad \text{for } x \neq 0 \\
y\text{-axis:} \quad & f(0,y) = \frac{0 \cdot y^2}{0^4 + y^4} = 0 \quad \text{for } y \neq 0 \\
y = kx\text{:} \quad & f(x,kx) = \frac{k^2x^4}{x^4 + k^4x^4} = \frac{k^2}{1+k^4}
\end{aligned}$$

In all three cases we obtain a limiting value for $f(x,y)$ as (x,y) approaches $(0,0)$ along a straight line, namely the limit is 0 in the first two cases and $k^2/(1+k^4)$ in the third case. However, in order for $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ to exist, we must get the *same* limit no matter how (x,y) approaches $(0,0)$. This does not happen in the present case because along the line $y = kx$ we get a different limit, $k^2/(1+k^4)$, from the limit we get along the axes. (For example if $k = 1$ we get the limit $1/2$, which is different from 0.)

A2. Consider the function

$$f(x,y) = \begin{cases} x^3y^2/(x^4+y^4) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

This function is continuous at $(0,0)$ but we are not asking you to show this. Instead, show that $f(x,y)$ is not differentiable at $(0,0)$ by investigating what happens along the coordinate axes and along the lines $y = kx$. (The graph of a differentiable function must have a unique tangent plane at each point, containing *all* the tangent lines to the graph at that point.)

Solution. We begin as in the previous problem, evaluating $f(x,y)$ along lines through

$(0,0)$:

$$\begin{aligned}
&x\text{-axis:}\quad f(x,0)=\frac{x^3\cdot 0}{x^4+0^4}=0\quad\text{for } x\neq 0\\
&y\text{-axis:}\quad f(0,y)=\frac{0\cdot y^2}{0^4+y^4}=0\quad\text{for } y\neq 0\\
&y=kx\text{:}\quad f(x,kx)=\frac{k^2x^5}{x^4+k^4x^4}=\left(\frac{k^2}{1+k^4}\right)x
\end{aligned}$$

The first two cases show that the graph of $f(x,y)$ contains the x-axis and the y-axis. This implies that the tangent plane to the graph at the origin must be the xy-plane since this is the only plane containing both the x and y axes. However in the third case (the lines $y=kx$) the function $\left(\frac{k^2}{1+k^4}\right)x$ has nonzero slope if $k\neq 0$, so the portion of the graph of $f(x,y)$ lying over the line $y=kx$ has its tangent line not contained in the xy-plane. This means the graph of $f(x,y)$ does not have a unique tangent plane at the origin, so the function $f(x,y)$ cannot be differentiable at the origin.

Remark. In case you are curious, here is an argument for showing that $f(x,y)$ is continuous at the origin. Using one-variable calculus we see that the function $g(k)=\frac{k^2}{1+k^4}$ approaches 0 as k approaches ∞ or $-\infty$. This implies that $g(k)$ has a maximum value c which is finite. Thus $\frac{k^2}{1+k^4}\leq c$ for all k. It follows that $|f(x,y)|\leq c|x|$ for all x and y. Hence $f(x,y)$ approaches 0 as x approaches 0. In particular this is true as (x,y) approaches $(0,0)$, which says $f(x,y)$ is continuous at $(0,0)$ if we define $f(0,0)=0$.