

Problems from the book:

Section 8.2:

5. We are given the vector field $\mathbf{F} = (zx + z^2y + x)\mathbf{i} + (z^3yx + y)\mathbf{j} + z^4x^2\mathbf{k}$. To compute the integral of $\text{curl}(\mathbf{F})$ over a surface S it suffices (by Stokes' theorem) to compute the line integral of \mathbf{F} over the boundary of S , i.e., the total circulation of \mathbf{F} around the boundary of S . The surface we are given has boundary equal to the unit circle $x^2 + y^2 = 1$ in the xy -plane. This circle has $z = 0$ since it is in the xy -plane, and when $z = 0$ the vector field \mathbf{F} simplifies to $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, the standard radial vector field in the xy -plane. This vector field is perpendicular to the circle, so $\mathbf{F} \cdot \mathbf{T} = 0$ for \mathbf{T} the unit tangent vector to the circle. Thus the line integral $\int_{\partial S} \mathbf{F} \cdot \mathbf{T} ds = 0$.

Another way to arrive at the same conclusion is just to compute the line integral directly. The line integral is $\int x dx + y dy$, and the circle is parametrized by $x = \cos t$, $y = \sin t$, and so $dx = -\sin t dt$ and $dy = \cos t dt$, hence $x dx + y dy = 0$.

Yet another way to do this is to note that the vector field $x\mathbf{i} + y\mathbf{j}$ is the gradient field of $f(x, y) = \frac{1}{2}(x^2 + y^2)$ which implies that the line integral around any closed curve is zero.

6. Let's first compute the curl of the vector field $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$. This is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (0 - 0)\mathbf{i} + (0 - 0)\mathbf{j} + (0 - 0)\mathbf{k} = 0$$

Thus $\iint_S \text{curl}(\mathbf{F}) dS = 0$. We can check this by computing $\int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$ where ∂S consists of the three line segments joining $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. We have $\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial S} yz dx + xz dy + xy dz$. Each of the three line segments of ∂S has either $x = 0$, $y = 0$, or $z = 0$, which makes $yz dx + xz dy + xy dz = 0$ so the line integral is zero.

Alternatively, note that \mathbf{F} is the gradient field of $f(x, y, z) = xyz$. This automatically implies that $\text{curl}(\mathbf{F}) = 0$, and it also implies that the line integral of \mathbf{F} around any closed curve is zero.

19a. Let $\mathbf{v} = (a, b, c)$, a fixed vector. Then the vector field $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ has $\text{curl}(\mathbf{F}) = 0$ hence by Stokes' theorem the line integral of \mathbf{F} around the boundary ∂S of any surface S is zero. Alternatively, we can argue that \mathbf{F} is the gradient field of the function $f = ax + by + cz$, so the line integral of \mathbf{F} around any closed curve must be zero.

21. To verify Stokes' Theorem for the vector field $\mathbf{F} = (z, x, y)$ on the helicoid surface $\Phi(r, \theta) = (r \cos \theta, r \sin \theta, \theta)$, $0 \leq r \leq 1$, $0 \leq \theta \leq \pi/2$, we first compute the surface integral

$\iint_S \text{curl}(\mathbf{F}) \cdot (\Phi_r \times \Phi_\theta) dr d\theta$. We have

$$\begin{aligned}\Phi_r &= (\cos \theta, \sin \theta, 0) \\ \Phi_\theta &= (-r \sin \theta, r \cos \theta, 1) \\ \Phi_r \times \Phi_\theta &= (\sin \theta, -\cos \theta, r)\end{aligned}\quad \text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

Then we have

$$\begin{aligned}\iint_S \text{curl}(F) \cdot (\Phi_r \times \Phi_\theta) dr d\theta &= \int_0^{\pi/2} \int_0^1 (\sin \theta - \cos \theta + r) dr d\theta \\ &= \int_0^{\pi/2} \int_0^1 (\sin \theta - \cos \theta) dr d\theta + \int_0^{\pi/2} \int_0^1 r dr d\theta \\ &= \int_0^1 dr \int_0^{\pi/2} (\sin \theta - \cos \theta) d\theta + \int_0^1 r dr \int_0^{\pi/2} d\theta \\ &= (1)(0) + \left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi}{4}\end{aligned}$$

The other way to compute this is as the line integral $\int z dx + x dy + y dz$ over the boundary of S , which consists of three straight line segments and one curved segment of a helix. For the vertical segment we have $x = 0$ and $y = 0$ so the line integral is zero for this segment. For the segment in the x -axis we have $y = 0$ and $z = 0$ so the line integral is again zero. For the top horizontal segment we have $x = 0$ and $z = \theta = \pi/2$, a constant, so $dz = 0$ and again the line integral is zero. Finally for the curved segment, this is where $r = 1$ so we have the parametrization $(x, y, z) = (\cos \theta, \sin \theta, \theta)$, $0 \leq \theta \leq \pi/2$, and the line integral is

$$\int_{\partial S} z dx + x dy + y dz = \int_0^{\pi/2} [\theta(-\sin \theta) + \cos^2 \theta + \sin \theta] d\theta$$

Using integration by parts for the first term and the identity $\cos^2 \theta = (1 + \cos 2\theta)/2$ for the second term, we get

$$\theta \cos \theta - \sin \theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} - \cos \theta \Big|_0^{\pi/2} = -1 + \frac{\pi}{4} + 1 = \frac{\pi}{4}$$

Section 8.3:

3. If $\nabla f = (2xyz + \sin x)\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ then

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2xyz + \sin x \implies f = x^2yz - \cos x + g(y, z) \\ \frac{\partial f}{\partial y} &= x^2z \implies f = x^2yz + h(x, z) \\ \frac{\partial f}{\partial z} &= x^2y \implies f = x^2yz + k(x, y)\end{aligned}$$

These conditions are all satisfied if we take $f = x^2yz - \cos x$.

4. We can use the formula $\int_C \nabla f \cdot ds = f(B) - f(A)$ for C any path from A to B . In the present case we have $f = x^2yz - \cos x$ and C goes from $A = (1, 0, 0)$ to $B = (-1, 0, \pi^4)$ so the answer is

$$x^2yz - \cos x \Big|_{(1,0,0)}^{(-1,0,\pi^4)} = -\cos(-1) + \cos(1) = 0$$

5. The vector field $\mathbf{F}(x, y) = f(x)\mathbf{i} + f(y)\mathbf{j}$ is the gradient of the function $g(x) + g(y)$ where g is an antiderivative of f , i.e., $g' = f$. (In fact this argument works more generally to show that $f_1(x)\mathbf{i} + f_2(y)\mathbf{j}$ is the gradient of $g_1(x) + g_2(y)$ where $g_1' = f_1$ and $g_2' = f_2$.)

7. If the vector field $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is a gradient field, then its curl must be zero. We compute:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y & z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - x\mathbf{k} = -x\mathbf{k}$$

Since this is not zero, \mathbf{F} is not a gradient field.

10. To compute the circulation of $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ around the unit circle in the xy -plane, note that in the xy -plane we have $z = 0$ so $\mathbf{F} = xy\mathbf{i}$ in the xy -plane. The vectors $xy\mathbf{i}$ lie in the xy -plane, so computing the circulation of $xy\mathbf{i}$ around the unit circle is really a 2-dimensional problem. Using Green's theorem, this is equivalent to computing the total curl in the disk $x^2 + y^2 \leq 1$. The curl of $xy\mathbf{i}$ is $-x$, so we need to compute $\iint_D -x \, dx \, dy$ for D the unit disk. By symmetry this integral will be zero since reflection across the y -axis takes D to itself and changes the sign of $-x$.

13. (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ is obviously the gradient of $f(x, y) = \frac{1}{2}(x^2 + y^2)$. (This is a special case of exercise 5.)

(b) The (scalar) curl of $\mathbf{F}(x, y) = xy\mathbf{i} + xy\mathbf{j}$ is $y - x$, which is nonzero, so \mathbf{F} is not a gradient field.

(c) The curl of $\mathbf{F} = (x^2 + y^2)\mathbf{i} + 2xy\mathbf{j}$ is 0 and \mathbf{F} is the gradient of $f(x, y) = \frac{1}{3}x^3 + xy^2$.

15. (a) To find a function $f(x, y)$ with $\nabla f = (xy^2 + 3x^2y)\mathbf{i} + (x^3 + x^2y)\mathbf{j}$ we solve:

$$\begin{aligned} \frac{\partial f}{\partial x} = xy^2 + 3x^2y &\implies f = \frac{1}{2}x^2y^2 + x^3y + g(y) \\ \frac{\partial f}{\partial y} = x^3 + x^2y &\implies f = x^3y + \frac{1}{2}x^2y^2 + h(x) \end{aligned}$$

So $f = \frac{1}{2}x^2y^2 + x^3y$ works. The line integral of \mathbf{F} over any path from $(1, 1)$ to $(3, 0)$ is

then

$$\frac{1}{2}x^2y^2 + x^3y \Big|_{(1,1)}^{(3,0)} = -\frac{3}{2}$$

(b) This time we solve

$$\frac{\partial f}{\partial x} = 2x(y^2 + 1)^{-1} \implies f = x^2(y^2 + 1)^{-1} + g(y)$$

$$\frac{\partial f}{\partial y} = -2y(x^2 + 1)(y^2 + 1)^{-2} \implies f = (x^2 + 1)(y^2 + 1)^{-1} + h(x)$$

This are simultaneously satisfied if we take $f = (x^2 + 1)(y^2 + 1)^{-1}$. The given curve goes from $(-1, 0)$ to $(0, 0)$ so the line integral equals

$$\frac{x^2 + 1}{y^2 + 1} \Big|_{(-1,0)}^{(0,0)} = 1 - 2 = -1$$

(c) In this case we have

$$\frac{\partial f}{\partial x} = \cos(xy^2) - xy^2 \sin(xy^2)$$

$$\frac{\partial f}{\partial y} = -2x^2y \sin(xy^2)$$

The second equation is simpler, so let's attack it first. We want an antiderivative of $-2x^2y \sin(xy^2)$ with respect to y . Note that the function $\cos(xy^2)$ has its partial with respect to y the function $-2xy \sin(xy^2)$, so if we take $f = x \cos(xy^2)$ then $\frac{\partial f}{\partial y}$ is what we want it to be, $-2x^2y \sin(xy^2)$. For this function f , if we compute $\frac{\partial f}{\partial x}$ we get $\sin(xy^2) - 2x^2y \cos(xy^2)$, so it works to take $f = x \cos(xy^2)$. For the line integral over a path from $(e^{-1}, 1)$ to $(1, e)$ we get

$$x \cos(xy^2) \Big|_{(e^{-1},1)}^{(1,e)} = \cos(e^2) - e^{-1} \cos(e^{-1})$$

Additional Problems

A1. Use Stokes' theorem to compute the total circulation of the vector field $\mathbf{F}(x, y, z) = -y \mathbf{i} + x \mathbf{j} + yz \mathbf{k}$ about the closed curve formed by the intersection of the cylinder $x^2 + y^2 = 1$ and the surface $z = x^2 - y^2$.

Solution. By Stokes' theorem we can compute the circulation as the surface integral of $\text{curl}(\mathbf{F})$ over the portion of the surface $z = x^2 - y^2$ inside the cylinder $x^2 + y^2 = 1$. For the curl we have

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & yz \end{vmatrix} = z \mathbf{i} + 2 \mathbf{k}$$

The surface is part of the graph $z = g(x, y) = x^2 - y^2$ so $d\mathbf{S} = (-g_x, -g_y, 1) dx dy = (-2x, 2y, 1) dx dy$. The surface integral is then

$$\iint_D \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_D (z, 0, 2) \cdot (-2x, 2y, 1) dx dy = \iint_D [-2x(x^2 - y^2) + 2] dx dy$$

where D is the unit disk in the xy -plane. By symmetry the integral of $-2x(x^2 - y^2)$ will be zero since replacing x by $-x$ changes the sign of $-2x(x^2 - y^2)$. Thus we are left with $\iint_D 2 dx dy = 2 \operatorname{Area}(D) = 2\pi$.

A2. Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y, z) = xy \mathbf{i} + 2z \mathbf{j} + 2y \mathbf{k}$, and C is the curve of intersection of the plane $x + z = 5$ and the cylinder $x^2 + y^2 = 5$.

Solution. This is very similar to the preceding problem. We have

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2z & 2y \end{vmatrix} = -x \mathbf{k}$$

The surface is the graph $z = g(x, y) = 5 - x$ so $d\mathbf{S} = (1, 0, 1) dx dy$ and the surface integral of the curl is $\iint_D -x dx dy$ where D is the disk $x^2 + y^2 \leq 5$. The value of the integral is zero by symmetry.

A3. If S is an ellipsoid and \mathbf{F} satisfies the hypotheses of Stokes' theorem, what is the value of $\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$? (Give reasons for your answer.)

Solution. The answer is that $\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$ is zero. This is true not just for an ellipsoid, but for any closed surface (i.e., surface that completely encloses a finite region of space). One argument that could be given is that by Stokes' theorem we have $\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$ and ∂S is empty so the line integral is zero. If this seems too mystical, one could also argue in the following way. Choose a closed curve C in S that separates S into two subsurfaces S_1 and S_2 whose common boundary is C . The orientation that C inherits from S_1 by the usual "surface on the left" rule is the opposite of the orientation inherited from S_2 , because if you walk along C with S_1 on your left then S_2 is on your right. Thus by applying Stokes' theorem to S_1 and S_2 separately we get

$$\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_{S_1} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} + \iint_{S_2} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{s} - \int_C \mathbf{F} \cdot d\mathbf{s} = 0$$

A4. Let $\mathbf{F}(x, y, z) = (ax^3 - 3xz^2) \mathbf{i} + (x^2y + by^3) \mathbf{j} + (cz^3) \mathbf{k}$. Let C be a fixed closed curve and consider all possible smooth surfaces S whose boundary curve is C . Find the values of a , b , and c for which $\iint_S \mathbf{F} \cdot d\mathbf{S}$ is independent of the choice of the surface S .

Solution. If the surface integral is independent of the choice of S , then for any solid region R whose boundary is formed by two choices S_1 and S_2 for S , we can conclude that $\iint_{\partial R} \mathbf{F} \cdot d\mathbf{S} = 0$ since this is saying that the total flux of F across ∂R is zero, which is true because the outward flux across one of S_1 and S_2 equals the inward flux across the other. By Gauss's theorem we deduce that $\iiint_R \operatorname{div}(\mathbf{F}) = 0$. Since this holds for all choices of S_1 and S_2 , it follows that $\operatorname{div}(\mathbf{F})$ itself must be zero. The vector field \mathbf{F} that we are given has $\operatorname{div}(\mathbf{F}) = 3ax^2 - 3z^2 + x^2 + 3by^2 + 3cz^2 = (3a + 1)x^2 + 3by^2 + (3c - 3)z^2$. For this to be zero for all x, y, z we must have $a = -1/3$, $b = 0$, and $c = 1$.