Problems from the book:

Section 7.3:

1. First we compute a normal vector $\Phi_u \times \Phi_v$ for the surface parametrized by $\Phi(u,v) = (2u, u^2+v, v^2)$. We have

$$\begin{aligned} \Phi_u &= (2, 2u, 0) \\ \Phi_v &= (0, 1, 2v) \\ \text{so}\quad \Phi_u \times \Phi_v &= (4uv, -4v, 2) \end{aligned}$$

The given point $(0,1,1)$ corresponds to $(u,v) = (0,1)$ so the normal vector here is $(0,-4,2)$. The tangent plane is then $0(x-0) - 4(y-1) + 2(z-1) = 0$, or $z = 2y-1$.

15. (a) The hint was to follow Example 5 on p.458, but the answer in the back of the book indicates another approach using cylindrical coordinates, so we will follow this idea. In cylindrical coordinates the equation of the hyperboloid $x^2+y^2-z^2 = 25$ becomes $r^2 - z^2 = 25$, or $r = \sqrt{z^2+25}$. We can use θ and z as the parameters, and then $\Phi(\theta, z) = (x,y,z) = (\sqrt{z^2+25}\,\cos\theta, \sqrt{z^2+25}\,\sin\theta, z)$.

(b) For the normal vector we compute

$$\begin{aligned} \Phi_\theta &= (-\sqrt{z^2+25}\,\sin\theta, \sqrt{z^2+25}\,\cos\theta, 0) \\ \Phi_z &= (z(z^2+25)^{-1/2}\cos\theta, z(z^2+25)^{-1/2}\sin\theta, 1) \\ \Phi_\theta \times \Phi_z &= (\sqrt{z^2+25}\,\cos\theta, \sqrt{z^2+25}\,\sin\theta, -z) \end{aligned}$$

To make this a unit vector we divide by its length which is $\sqrt{2z^2+25}$ to get

$$(\sqrt{z^2+25}\,\cos\theta, \sqrt{z^2+25}\,\sin\theta, -z)/\sqrt{2z^2+25}$$

(c) A point $(x_0, y_0, 0)$ on the hyperboloid has $z = 0$ so this point has the form $\Phi(\theta_0, 0) = (5\cos\theta_0, 5\sin\theta_0, 0)$. The normal vector $\Phi_\theta \times \Phi_z$ at this point is $(5\cos\theta_0, 5\sin\theta_0, 0)$ which happens to equal $(x_0, y_0, 0)$. The tangent plane here is $x_0(x-x_0) + y_0(y-y_0) + 0(z) = 0$, or $x_0 x + y_0 y = x_0^2 + y_0^2 = 25$.

(d) The lines $(x,y,z) = (x_0 \pm t y_0, y_0 \mp t x_0, 5t)$ lie on the surface because they satisfy the equation $x^2+y^2-z^2 = 25$, namely $(x_0 \pm t y_0)^2 + (y_0 \mp t x_0)^2 - (5t)^2 = x_0^2 \pm 2x_0 t y_0 + t^2 y_0^2 + x_0^2 \mp 2x_0 t y_0 + t^2 y_0^2 - 25t^2 = x_0^2 + y_0^2 + t^2(x_0^2 + y_0^2) - 25t^2$, and we are assuming that $x_0^2 + y_0^2 = 25$, so we end up with 25. Thus these two lines lie in the surface, so it's automatically true that they lie in the tangent plane at the point of tangency $(x_0, y_0, 0)$.

Section 7.4:

5. We are given $\Phi(u,v)=(u-v,u+v,uv)$ and from this we compute

$$\begin{aligned}\Phi_u &= (1,1,v)\\ \Phi_v &= (-1,1,u)\\ \Phi_u\times\Phi_v &= (u-v,-u-v,2)\\ |\Phi_u\times\Phi_v| &= \sqrt{2u^2+2v^2+4}\end{aligned}$$

The surface area is then $\iint_D \sqrt{2u^2+2v^2+4}\,du\,dv$ with D the unit disk, so we switch to polar coordinates to get

$$\int_0^{2\pi}\int_0^1 r(2r^2+4)^{1/2}\,dr\,d\theta = 2\pi\Big(\frac{1}{6}\Big)\big(2r^2+4\big)^{3/2}\Big|_0^1 = \frac{\pi}{3}\big(6\sqrt{6}-8\big)$$

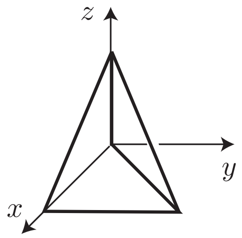
6. Using spherical coordinates, the portion of the sphere is parametrized by $\Phi(u,v)=(\sin u\cos v,\sin u\sin v,\cos u)$ for $0\le u\le\pi/4$ and $0\le v\le 2\pi$. We have

$$\begin{aligned}\Phi_u &= (\cos u\cos v,\cos u\sin v,-\sin u)\\ \Phi_v &= (-\sin u\sin v,\sin u\cos v,0)\\ \Phi_u\times\Phi_v &= (\sin^2 u\cos v,\sin^2 u\sin v,\sin u\cos u)\\ |\Phi_u\times\Phi_v| &= \sin u\end{aligned}$$

The surface area is then

$$\int_0^{2\pi}\int_0^{\pi/4}\sin u\,du\,dv = 2\pi\big(-\cos u\big)\Big|_0^{\pi/4} = 2\pi\Big(-\frac{1}{\sqrt{2}}+1\Big) = \pi\big(2-\sqrt{2}\big)$$

Section 7.5:



1. We compute $\iint_S xy\,dS$ over the four sides of the tetrahedron separately and then add the results. For the side in the xz-plane we have $y=0$ so the integral is $\iint_S 0\,dS=0$. For the other three sides we will use the fact that for the graph of a function $g(u,v)$ we have $dS=|T_u\times T_v|\,du\,dv=\sqrt{g_u^2+g_v^2+1}\,du\,dv$.

The bottom triangle can be regarded as the graph of the function $g(x,y)=0$ so $dS=1\cdot dx\,dy$ and we get $\int_0^1\int_0^x xy\,dy\,dx$ which works out to be $\frac{1}{8}$. For the top triangle which is defined by the equation $x+z=1$ we can regard this as the graph of $g(x,y)=z=1-x$

so $dS = \sqrt{2}\,dx\,dy$ and we get $\int_0^1\int_0^x xy\sqrt{2}\,dy\,dx$ so this is just $\sqrt{2}$ times the previous integral, or $\sqrt{2}/8$. For the right side we can regard this as the graph of $g(x,z) = x$, so $dS = \sqrt{2}\,dx\,dz$ and the integral is $\int_0^1\int_0^{1-x} x^2\sqrt{2}\,dz\,dx$ which works out to be $\sqrt{2}/12$. Adding up, we get $\frac{1}{8} + \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{12} = \frac{3+5\sqrt{2}}{24}$.

3. The upper hemisphere is the graph of $g(x,y) = \sqrt{a^2 - x^2 - y^2}$ so we get $dS = \sqrt{g_x^2 + g_y^2 + 1}\,dx\,dy$ which works out to be $a(a^2 - x^2 - y^2)^{-1/2}\,dx\,dy$. Plugging this into $\iint_D z\,dS$ with $z = (a^2 - x^2 - y^2)^{1/2}$ yields $\iint_D a\,dx\,dy$. This is just a times the area of D. The region D is $x^2 + y^2 \le a^2$, with area πa^2, so the final answer is πa^3.

8. To evaluate $\iint_S z^2\,dS$ over the cube with vertices $(\pm 1, \pm 1, \pm 1)$ we can use symmetry to say that the integral is the same on the top as on the bottom, and also the integral is the same on each of the four vertical sides, so the final answer will be two times the integral over the top plus four times the integral over one of the vertical sides. For the top we regard this as the graph of the function $g(x,y) = 1$ so $dS = \sqrt{g_x^2 + g_y^2 + 1}\,dx\,dy = dx\,dy$ and the integral becomes $\iint_S 1\,dx\,dy$ which is just the area of this side of the cube, 4. For one of the vertical sides, say the side $x = 1$, we view this as the graph of $g(y,z) = 1$ so $dS = \sqrt{g_y^2 + g_z^2 + 1}\,dy\,dz = dy\,dz$ and the integral is $\int_{-1}^1\int_{-1}^1 z^2\,dy\,dz = \int_{-1}^1 dy\int_{-1}^1 z^2\,dz = (2)(\frac{2}{3}) = \frac{4}{3}$. Adding up for all the sides, we get $2(4) + 4(\frac{4}{3}) = 40/3$.

Section 7.6:

3. We can compute the integral of $\mathbf{E}(x,y,z) = (2x, 2y, 2z)$ over the upper hemisphere of $x^2 + y^2 + z^2 = 1$ by using the parametrization in problem 6 in 7.4, above. Thus $\Phi(u,v) = (\sin u\cos v, \sin u\sin v, \cos u)$ and $\Phi_u \times \Phi_v = (\sin^2 u\cos v, \sin^2 u\sin v, \sin u\cos u)$. To compute $\iint_S \mathbf{E}\cdot(\Phi_u \times \Phi_v)\,du\,dv$ over the hemisphere we first compute $\mathbf{E}\cdot(\Phi_u \times \Phi_v)$. This is

$$(2\sin u\cos v, 2\sin u\sin v, 2\cos u)\cdot(\sin^2 u\cos v, \sin^2 u\sin v, \sin u\cos u)$$

which works out to be just $2\sin u$. Thus for the hemisphere we have

$$\iint_S \mathbf{E}\cdot(\Phi_u \times \Phi_v)\,du\,dv = \int_0^{2\pi}\int_0^{\pi/2} 2\sin u\,du\,dv = 2\pi\Big(-2\cos u\,\Big|_0^{\pi/2}\Big) = 2\pi(2) = 4\pi$$

For the disk $x^2 + y^2 \le 1$ in the xy-plane we have $z = 0$ so $\mathbf{E}(x,y,0) = (2x, 2y, 0)$. The unit normal vector to this disk will be $\mathbf{n} = (0, 0, \pm 1)$ so $\mathbf{E}\cdot\mathbf{n} = 0$ and the flux over this face will be $\iint_D \mathbf{E}\cdot\mathbf{n}\,dS = 0$. Thus our final answer is 4π.

4. The surface $x^2+z^2=1$, $0\le y\le 1$, $0\le x\le 1$ is half of a cylinder. We can computer the flux of $\mathbf{F}=\sqrt{y}\,\mathbf{i}$ across this half-cylinder in two ways, either by parametrizing the surface using cylindrical coordinates (with y and z interchanged from what they usually are in cylindrical coordinates) or by regarding the surface as the graph of the function $x=g(y,z)=\sqrt{1-z^2}$. For practice, we will do it both ways.

The first way is to parametrize the half-cylinder by $\Phi(u,v)=(\cos u,v,\sin u)$, $-\frac{\pi}{2}\le u\le\frac{\pi}{2}$, $0\le v\le 1$. Then $\Phi_u=(-\sin u,0,\cos u)$ and $\Phi_v=(0,1,0)$ so $\Phi_u\times\Phi_v=(-\cos u,0,-\sin u)$. This is oriented toward the inside of the cylinder, so to get the more usual outward orientation we will replace it by $(\cos u,0,\sin u)$. Taking the dot product with the vector field $\mathbf{F}=\sqrt{y}\,\mathbf{i}=\sqrt{v}\,\mathbf{i}$ we get $\sqrt{v}\,\cos u$. Thus the flux is

$$\int_0^1\int_{-\pi/2}^{\pi/2}\sqrt{v}\,\cos u\,du\,dv=\Big(\sin u\,\Big|_{-\pi/2}^{\pi/2}\Big)\Big(\frac{2v^{3/2}}{3}\,\Big|_0^1\Big)=2\Big(\frac{2}{3}\Big)=\frac{4}{3}$$

Now let's do the problem by treating the half-cylinder as the graph of the function $x=g(y,z)=\sqrt{1-z^2}$ for $0\le y\le 1$, $-1\le z\le 1$. The normal vector is then $\mathbf{T}_y\times\mathbf{T}_z=(1,-g_y,-g_z)$. Taking the dot product of this with $\mathbf{F}=\sqrt{y}\,\mathbf{i}$ gives $\sqrt{y}$. Thus the flux equals

$$\int_{-1}^1\int_0^1\mathbf{F}\cdot(\mathbf{T}_y\times\mathbf{T}_z)\,dy\,dz=\int_{-1}^1\int_0^1 y^{1/2}\,dy\,dz=\Big(\frac{2y^{3/2}}{3}\,\Big|_0^1\Big)\Big(z\,\Big|_{-1}^1\Big)=\Big(\frac{2}{3}\Big)\Big(2\Big)=\frac{4}{3}$$

16. This could be done either by parametrizing the cone (using cylindrical coordinates) or by regarding the cone as the graph of the function $z=g(x,y)=\sqrt{x^2+y^2}$. Here's how to do it as the graph. The normal vector is

$$\mathbf{T}_x\times\mathbf{T}_y=(-g_x,-g_y,1)=(-x/\sqrt{x^2+y^2},-y/\sqrt{x^2+y^2},1)$$

In case (a) we have $\mathbf{F}=(0,0,-1)$ so $\mathbf{F}\cdot(\mathbf{T}_x\times\mathbf{T}_y)=-1$ and the flux is $\iint_D -dx\,dy$, which is -1 times the area of D, the unit disk, so the answer is $-\pi$. The minus sign is due to the fact that the flow is downward while the normal vector to the graph $z=g(x,y)$ is always upward since its z-coordinate is 1.

In case (b) we have $\mathbf{F}=-\frac{1}{\sqrt{2}}(1,0,1)$ so we have flux

$$\iint_D\mathbf{F}\cdot(\mathbf{T}_x\times\mathbf{T}_y)\,dx\,dy=-\frac{1}{\sqrt{2}}\iint_D\Big(\frac{-x}{\sqrt{x^2+y^2}}+1\Big)\,dx\,dy$$

This could be computed directly, but notice that the integral of the term $-x/\sqrt{x^2+y^2}$ will be zero by symmetry, so we are left with $-\frac{1}{\sqrt{2}}\iint_D dx\,dy=-\pi/\sqrt{2}$

Additional Problems

A1. Find the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z)=(xy,yz,xz)$ and S is the part of the paraboloid $z=4-x^2-y^2$ that lies above the square $0\le x\le 1,\ \ 0\le y\le 1$, and S has the upward normal orientation.

Solution. The surface is the graph of $g(x,y)=4-x^2-y^2$, so we have $T_x\times T_y=(2x,2y,1)$. Thus $\mathbf{F}\cdot d\mathbf{S}=(xy,yz,xz)\cdot(2x,2y,1)=2x^2y+2y^2z+xz$. Therefore the surface integral is

$$\begin{aligned}\iint \mathbf{F}\cdot d\mathbf{S} &= \int_0^1\int_0^1 [2x^2y+2y^2(4-x^2-y^2)+x(4-x^2-y^2)]\,dy\,dx\\ &= \int_0^1 (x^2/3+11x/3-x^3+34/15)\,dx = 713/180\end{aligned}$$

A2. Find the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z)=x^2\,\mathbf{i}+y^2\,\mathbf{j}+z^2\,\mathbf{k}$ and S is the complete boundary surface of the solid half-cylinder $0\le z\le\sqrt{1-y^2},\ \ 0\le x\le 2$.

Solution. Let us divide the surface into four part : (i) S_1, the top surface (part of the circular cylinder $x^2+y^2=1$), (ii) S_2, the bottom surface (part of the xy-plane), (iii) S_3, the front half-disk in the plane $x=2$, and (iv) S_4, the back half-disk in the plane $x=0$.

(i) On S_1: The surface is the graph of $g(x,y)=\sqrt{1-y^2}$ for $0\le x\le 2,\ -1\le y\le 1$ with upward orientation, so $T_x\times T_y=(0,y/\sqrt{1-y^2},1)$. Thus

$$\iint_{S_1}\mathbf{F}\cdot d\mathbf{S}=\int_0^2\int_{-1}^1\Big(\frac{y^3}{\sqrt{1-y^2}}+z^2\Big)\,dy\,dx=\int_0^2\int_{-1}^1\Big(\frac{y^3}{\sqrt{1-y^2}}+1-y^2\Big)\,dy\,dx$$

Notice that the integral of the term $y^3/\sqrt{1-y^2}$ will be zero by symmetry in the y direction. This leaves the integral of $1-y^2$ which easily works out to be $8/3$.

(ii) On S_2 : $z=0$ with $\mathbf{n}=(0,0,-1)$. Thus, $\mathbf{F}\cdot\mathbf{n}=-z^2=0$ on S_2, which implies that $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S}=0$.

(iv) Similarly, on S_4, $x=0$ with $\mathbf{n}=(-1,0,0)$. Thus $\mathbf{F}\cdot\mathbf{n}=-x^2=0$, which implies that $\iint_{S_4}\mathbf{F}\cdot d\mathbf{S}=0$.

(iii) On S_3: The surface is $x=2$ for $-1\le y\le 1,\ 0\le z\le\sqrt{1-y^2}$, oriented in the positive x-direction. Regarding y and z as parameters, we have $T_y\times T_z=\mathbf{i}$. Therefore,

$$\iint_{S_3}\mathbf{F}\cdot d\mathbf{S}=\int_{-1}^1\int_0^{\sqrt{1-y^2}}x^2\,dz\,dy=\int_{-1}^1\int_0^{\sqrt{1-y^2}}4\,dz\,dy=4\,Area(S_3)=2\pi$$

The final answer is thus $\frac{8}{3}+2\pi$.

A3. Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2+y^2}$, $1 \le z \le 4$ if the density function is $\delta(x,y,z) = 10 - z$.

Solution.

S is given by $\Phi(x,y) = (x, y, \sqrt{x^2+y^2})$, thus $||T_x \times T_y|| = \sqrt{1 + (x^2+y^2)/(x^2+y^2)} = \sqrt{2}$. The total mass is then

$$\begin{aligned} M &= \iint_S (10 - \sqrt{x^2+y^2})\, dS = \iint_S (10 - \sqrt{x^2+y^2})\sqrt{2}\, dA \\ &= \int_0^{2\pi}\int_1^4 \sqrt{2}(10-r)\, r\, dr\, d\theta = 2\pi\sqrt{2}\left(5r^2 - \frac{r^3}{3}\right)\Big|_1^4 = 108\sqrt{2}\,\pi \end{aligned}$$

A4. Use Gauss' Law (p.493) to find the charge enclosed by the cube whose vertices are the points $(\pm1, \pm1, \pm1)$ if the electric field is $\mathbf{E}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

Solution. Since the electric field is symmetric with respect to permutations of the x, y, z axes, and the surface is a cube which is also symmetric with respect to the permutations of x, y, z, the surface integral of $\mathbf{E}$ over S is 6 times the surface integral over one of the sides. Let us take the top side S_1. On the top we have $\mathbf{E} = (x, y, 1)$ and $\mathbf{n} = (0, 0, 1)$, thus $\mathbf{E} \cdot \mathbf{n} = 1$. Therefore $\iint_{S_1} \mathbf{E} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} dx\, dy = 4$. Then the charge enclosed by the cube is $Q = 6 \cdot 4 = 24$ (with respect to a suitable unit).