1. Let S be the level surface $f(x, y, z) = x^2 + 3y^2 + z^2 + xz = 6$.

(5 pts) (a) Find the tangent plane to S at the point $(1, 1, 1)$.

Solution. We have $\nabla f = (2x + z, 6y, 2z + x)$, and at $(1, 1, 1)$ this equals $(3, 6, 3)$. The tangent plane is therefore $3(x - 1) + 6(y - 1) + 3(z - 1) = 0$. This can be simplified to $x + 2y + z = 4$ if you want.

(5 pts) (b) Find all points on S where the tangent plane is horizontal (parallel to the xy-plane).

Solution. The tangent plane is horizontal where the normal vector ∇f is vertical, i.e., of the form $(0, 0, z)$. Thus we have the conditions $\frac{\partial f}{\partial x} = 2x + z = 0$ and $\frac{\partial f}{\partial y} = 6y = 0$, so $y = 0$ and $z = -2x$. Plugging $y = 0$ and $z = -2x$ into $x^2 + 3y^2 + z^2 + xz = 6$ gives $x^2 + 4x^2 - 2x^2 = 6$, which has the solutions $x = \pm\sqrt{2}$. Since $y = 0$ and $z = -2x$ we get the two points $\pm(\sqrt{2}, 0, -2\sqrt{2})$ where the tangent plane is horizontal.

(5 pts) (c) Show that the points on S where the tangent plane is vertical (parallel to a plane containing the z-axis) form an ellipse in some sloping plane.

Solution. Vertical tangent planes occur where ∇f is horizontal, of the form $(x, y, 0)$ so we get the condition $\frac{\partial f}{\partial z} = 0$. This gives the plane $x + 2z = 0$, or $x = -2z$. Plugging $x = -2z$ into the equation $x^2 + 3y^2 + z^2 + xz = 6$ gives $4z^2 + 3y^2 + z^2 - 2z^2 = 3y^2 + 3z^2 = 6$, or $y^2 + z^2 = 2$. This is a cylinder (with axis the x-axis), and intersecting this cylinder with the plane $x = -2z$ gives an ellipse in this plane.

2. Let $f(x, y) = \dfrac{x^2 + y^2}{2x}$.

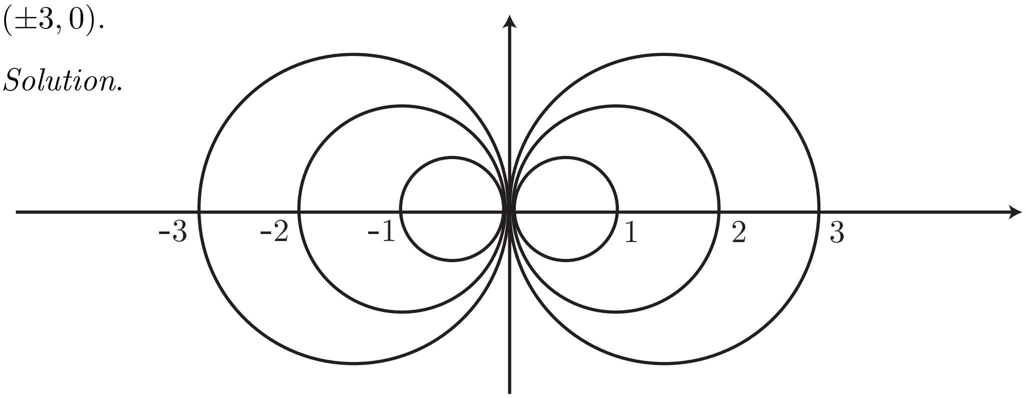
(5 pts) (a) Show that the level curves of f are circles passing through the origin.

Solution. A level curve is $f(x, y) = \dfrac{x^2 + y^2}{2x} = c$. This equation can be rewritten as $x^2 + y^2 = 2cx$ or $x^2 - 2cx + y^2 = 0$. Completing the square, this can be written as $(x - c)^2 + y^2 = c^2$, which is the equation of a circle of radius c and center at $(c, 0)$. This circle passes through the origin since its radius equals the distance of its center from the origin.

Alternatively, one can say that the earlier equation $x^2 - 2cx + y^2 = 0$ is a circle since it is a quadratic equation and the coefficients of x^2 and y^2 are equal. It passes through the origin because $(x, y) = (0, 0)$ satisfies the equation.

(5 pts) (b) Draw a sketch showing the level curves that pass through the six points $(\pm 1, 0)$, $(\pm 2, 0)$, $(\pm 3, 0)$.

Solution.



(5 pts) (c) Using the level curves of f, determine whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, and give a reason for your answer.

Solution. The limit does not exist because as (x, y) approaches $(0, 0)$ along each of the circles in the preceding figure, the value of $f(x, y)$ is a different constant since these circles are level curves $f(x, y) = c$ for different values of c.

(20 pts) **3.** Find all critical points of $f(x, y) = \sin x \cos y$ in the range $-\pi < x < \pi$ and $-\pi < y < \pi$ (note that these are strict inequalities) and determine whether each critical point is a local minimum, local maximum, or saddle.

Solution. To find critical points we solve $f_x = \cos x \cos y = 0$ and $f_y = -\sin x \sin y = 0$. In the first equation, if $\cos x = 0$ then $x = \pm\frac{\pi}{2}$, and then the second equation has $\sin x = \sin(\pm\frac{\pi}{2}) \neq 0$ hence $\sin y = 0$, which implies $y = 0$. Thus we have the solutions $(\pm\frac{\pi}{2}, 0)$ when $\cos x = 0$ in the first equation. The only other possibility in the first equation is that $\cos y = 0$ so $y = \pm\frac{\pi}{2}$. The second equation then implies that $\sin x = 0$ so $x = 0$ and we have two more critical points $(0, \pm\frac{\pi}{2})$, in addition to the first two critical points $(\pm\frac{\pi}{2}, 0)$. Now we apply the second derivative test. The Hessian matrix is

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} -\sin x \cos y & -\cos x \sin y \\ -\cos x \sin y & -\sin x \cos y \end{bmatrix}$$

At the two critical points $(\pm\frac{\pi}{2}, 0)$ this matrix is $\begin{bmatrix} \mp 1 & 0 \\ 0 & \mp 1 \end{bmatrix}$ with positive determinant, so by looking at the sign of the first entry f_{xx} in the matrix we see that $(\frac{\pi}{2}, 0)$ is a local maximum and $(-\frac{\pi}{2}, 0)$ is a local minimum. At the critical points $(0, \pm\frac{\pi}{2})$ the Hessian matrix is $\begin{bmatrix} 0 & \mp 1 \\ \mp 1 & 0 \end{bmatrix}$ which has negative determinant so these two critical points are saddles.

(20 pts) **4.** Find the maximal volume of a rectangular box that is contained inside the ellipsoid $x^2+9y^2+4z^2=9$, where the edges of the box are assumed to be parallel to the coordinate axes. (You can assume that a box of maximum volume exists.)

Solution. Since we are seeking to maximize volume, we can always enlarge a box inside the ellipsoid until all its vertices are actually on the ellipsoid. Let (x,y,z) be the vertex in the first octant. The dimensions of the box are then $2x \times 2y \times 2z$ so its volume is $8xyz$. We will maximize the function $f(x,y,z)=xyz$ with the constraint $g(x,y,z)=x^2+9y^2+4z^2=9$.

We solve the Lagrange multiplier equation $\nabla f = \lambda \nabla g$. This gives three equations $yz = 2\lambda x$, $xz = 18\lambda y$, and $xy = 8\lambda z$, in addition to the constraint equation. We can assume x, y, and z are all greater than 0, so the three equations can be solved for λ, giving $\lambda = \dfrac{yz}{2x} = \dfrac{xz}{18y} = \dfrac{xy}{8z}$. The equation $\dfrac{yz}{2x} = \dfrac{xz}{18y}$ gives $9y^2 = x^2$, so $x = 3y$ (we can discard the solution $x=-3y$ since x,y,z are all positive). Similarly, the equation $\dfrac{xz}{18y} = \dfrac{xy}{8z}$ gives $2z = 3y$. Plugging into the constraint equation, we get $9y^2+9y^2+9y^2 = 9$ so $3y^2 = 1$ and $y = 1/\sqrt{3}$. Then $x = 3y = \sqrt{3}$ and $z = 3y/2 = \sqrt{3}/2$. The maximum volume is then $8xyz = 4\sqrt{3}$.

5. Suppose that for some differentiable function $f(x,y,z)$ we know that the maximum value of the directional derivatives $D_{\mathbf{u}}$ at the point $(1,1,1)$ is 2, and this maximum occurs in the direction of the vector $(1,2,2)$.

(8 pts) (a) From this information, compute $\nabla f(1,1,1)$.

Solution. The maximum value of the directional derivative is always $||\nabla f||$ and this occurs in the direction of ∇f. Thus ∇f is a scalar times $(1,2,2)$, i.e., $\nabla f = \lambda(1,2,2)$, and we have $2 = ||\nabla f|| = \lambda ||(1,2,2)|| = \lambda\sqrt{1^2+2^2+2^2} = 3\lambda$, hence $\lambda = \frac{2}{3}$ and $\nabla f = \frac{2}{3}(1,2,2) = (\frac{2}{3}, \frac{4}{3}, \frac{4}{3})$.

(7 pts) (b) Compute $D_{\mathbf{u}}(1,1,1)$ in the directions of the vectors $(2,1,-2)$ and $(1,1,0)$.

Solution. We use the general formula $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$ for unit vectors $\mathbf{u}$. The vector $(2,1,-2)$ normalizes to the unit vector $\mathbf{u} = (\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$ so in this case $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = (\frac{2}{3}, \frac{4}{3}, \frac{4}{3}) \cdot (\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}) = 0$. For $(1,1,0)$ we normalize to get $\mathbf{u} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ so $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = (\frac{2}{3}, \frac{4}{3}, \frac{4}{3}) \cdot (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) = \sqrt{2}$.

(15 pts) **6.** Suppose that a differentiable function $f(x,y)$ has $\dfrac{\partial f}{\partial x}(5,3) = 4$ and $\dfrac{\partial f}{\partial y}(5,3) = 6$. Suppose also that x and y are related to variables u and v by $x = u^2 + v^2$ and $y = u^2 - v^2$. Compute $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ at $(u,v) = (2,1)$.

Solution. Note that when $(u,v) = (2,1)$ we have $(x,y) = (2^2 + 1^2, 2^2 - 1^2) = (5,3)$. Now apply the chain rule and plug in these values for (x,y) and (u,v):

$$\begin{aligned}
\frac{\partial f}{\partial u}(2,1) &= \frac{\partial f}{\partial x}(5,3)\frac{\partial x}{\partial u}(2,1) + \frac{\partial f}{\partial y}(5,3)\frac{\partial y}{\partial u}(2,1) \\
&= (4)(4) + (6)(4) = 40 \quad \text{since} \quad \frac{\partial x}{\partial u} = 2u \quad \text{and} \quad \frac{\partial y}{\partial u} = 2u \\
\frac{\partial f}{\partial v}(2,1) &= \frac{\partial f}{\partial x}(5,3)\frac{\partial x}{\partial v}(2,1) + \frac{\partial f}{\partial y}(5,3)\frac{\partial y}{\partial v}(2,1) \\
&= (4)(2) + (6)(-2) = -4 \quad \text{since} \quad \frac{\partial x}{\partial v} = 2v \quad \text{and} \quad \frac{\partial y}{\partial v} = -2v
\end{aligned}$$