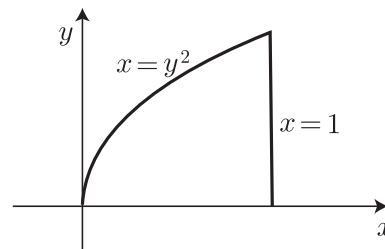


1. Compute  $\int_0^1 \int_{y^2}^1 4y \cos(x^2) dx dy$  by changing the order of integration. Include a sketch of the region of integration in your solution.

*Solution.*

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{x}} 4y \cos(x^2) dy dx &= \int_0^1 \left[ 2y^2 \cos(x^2) \Big|_0^{\sqrt{x}} \right] dx \\ &= \int_0^1 2x \cos(x^2) dx = \sin(x^2) \Big|_0^1 = \sin 1 \end{aligned}$$



2. Let  $R$  be the region in  $\mathbb{R}^3$  inside the sphere  $x^2 + y^2 + z^2 = 2$  and above the paraboloid  $z = x^2 + y^2$ .

(a) Sketch the region  $R$ .

(b) Set up the integral  $\iiint_R f(x, y, z) dV$  in rectangular coordinates, using any order of integration that you like. Include explicit limits of integration.

(c) Do the same as in part (b) using cylindrical coordinates.

(d) Do the same thing again using spherical coordinates.

*Solution.* To see where the two surfaces intersect, substitute  $x^2 + y^2 = z$  into  $x^2 + y^2 + z^2 = 2$  to get  $z + z^2 = 2$ . This has the solutions  $z = 1, -2$ , and we want the positive solution  $z = 1$ , so the two surfaces intersect where  $z = x^2 + y^2 = 1$ . The shadow of the region  $R$  in the  $xy$ -plane is the disk  $x^2 + y^2 \leq 1$ .

In rectangular coordinates the integral is then

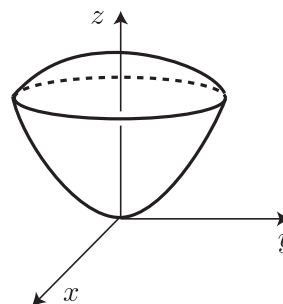
$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} f(x, y, z) dz dy dx$$

In cylindrical coordinates this becomes

$$\int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta$$

Spherical coordinates are trickier since one has to separate the region into two parts along the cone  $\phi = \pi/4$ . Above the cone the boundary of  $R$  is formed by the sphere  $\rho = \sqrt{2}$ , while below the cone the boundary of  $R$  is formed by the paraboloid  $z = x^2 + y^2$ , which in spherical coordinates becomes  $\rho \cos \phi = \rho^2 \sin^2 \phi$ . Solving this for  $\rho$  gives  $\rho = \cos \phi / \sin^2 \phi$ . Then the integral over the region  $R$  is the sum of the integrals over the two parts above and below the cone:

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} f(x, y, z) \rho^2 \sin \phi d\rho d\phi d\theta + \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{\cos \phi / \sin^2 \phi} f(x, y, z) \rho^2 \sin \phi d\rho d\phi d\theta$$



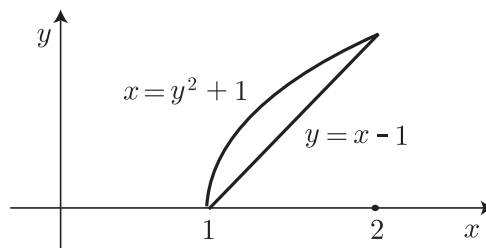
Here we should replace  $x$ ,  $y$ , and  $z$  in  $f(x, y, z)$  by their values in spherical coordinates,  $x = \rho \sin \phi \cos \theta$ , etc.

**3.** Let  $D$  be the region in the plane bounded by  $x = y^2 + 1$  and  $y = x - 1$ .

(a) Draw a sketch of  $D$  and set up double integrals for the area of  $D$  in both possible orders.

*Solution.*

$$\text{Area} = \int_1^2 \int_{x-1}^{\sqrt{x-1}} dy dx = \int_0^1 \int_{y^2+1}^{y+1} dx dy$$



(b) Compute the area of  $D$  by evaluating the integral from part (a) which you think is easiest.

*Solution.*

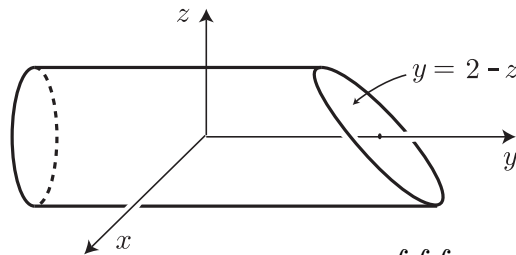
$$\int_0^1 \int_{y^2+1}^{y+1} dx dy = \int_0^1 \left[ x \Big|_{y^2+1}^{y+1} \right] dy = \int_0^1 (y - y^2) dy = \left( \frac{y^2}{2} - \frac{y^3}{3} \right) \Big|_0^1 = \frac{1}{6}$$

**4.** Let  $R$  be the region in  $\mathbb{R}^3$  which lies inside the cylinder  $x^2 + z^2 = 1$  and between the planes  $y = -2$  and  $y = 2 - z$ .

(a) Draw a sketch of  $R$  and set up the triple integral  $\iiint_R (x^2 + z^2) dy dz dx$  in the order shown. Be sure to include all limits of integration.

*Solution.*

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-2}^{2-z} (x^2 + z^2) dy dz dx$$



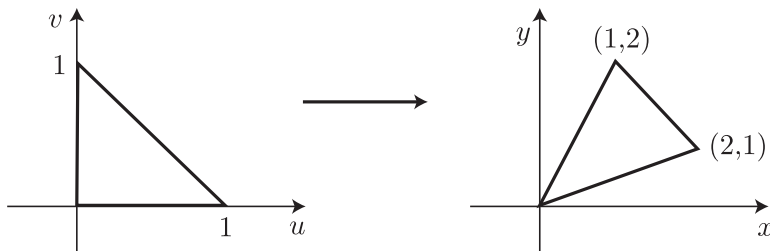
(b) By making an appropriate change of variables, compute the integral  $\iiint_R (x^2 + z^2) dV$ .

*Solution.* It is most convenient to use cylindrical coordinates but with the roles of  $y$  and  $z$  switched, so we have  $x = r \cos \theta$ ,  $y = y$ , and  $z = r \sin \theta$ . Then the integral becomes

$$\begin{aligned} \int_0^{2\pi} \int_0^1 \int_{-2}^{2-r \sin \theta} (r^2) r dy dr d\theta &= \int_0^{2\pi} \int_0^1 \left[ r^3 y \Big|_{-2}^{2-r \sin \theta} \right] dr d\theta \\ &= \int_0^{2\pi} \int_0^1 (4r^3 - r^4 \sin \theta) dr d\theta = \int_0^{2\pi} \left[ (r^4 - \frac{r^5 \sin \theta}{5}) \Big|_0^1 \right] d\theta \\ &= \int_0^{2\pi} \left( 1 - \frac{1}{5} \sin \theta \right) d\theta = \left( \theta + \frac{1}{5} \cos \theta \right) \Big|_0^{2\pi} = 2\pi \end{aligned}$$

5. (a) Find a linear change of variables mapping the triangle with vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(0, 1)$  to the triangle  $T$  bounded by the lines  $y = 2x$ ,  $y = x/2$ , and  $y = 3 - x$ .

*Solution.*



By solving the given equations two at a time one finds the vertices of  $T$  are at  $(0, 0)$ ,  $(2, 1)$ , and  $(1, 2)$ . Thus we want a linear transformation taking  $(1, 0)$  to  $(2, 1)$  and  $(0, 1)$  to  $(1, 2)$ .

The matrix of this transformation has  $(2, 1)$  and  $(1, 2)$  as its columns, so the matrix is  $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ . The linear transformation has the formula  $T(u, v) = (2u + v, u + 2v) = (x, y)$ .

(b) Use this change of variables to calculate  $\iint_T e^{x-2y} dx dy$ .

*Solution.* The Jacobian matrix is  $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ , which has determinant 3, so in  $uv$ -coordinates the integral becomes

$$\begin{aligned} \int_0^1 \int_0^{1-u} 3e^{-3v} dv du &= \int_0^1 \left[ -e^{-3v} \Big|_0^{1-u} \right] du = \int_0^1 (1 - e^{3u-3}) du = \left( u - \frac{e^{3u-3}}{3} \right) \Big|_0^1 \\ &= \frac{2 + e^{-3}}{3} \end{aligned}$$

6. Find the value of the triple integral  $\iiint_R (x^2 + y^2 + z^2) dV$  using spherical coordinates, where  $R$  is the part of the solid cone  $z \geq \sqrt{x^2 + y^2}$  that lies inside the sphere  $x^2 + y^2 + (z - 1)^2 = 1$ . (This sphere has radius 1 and center  $(0, 0, 1)$ , and it has a rather simple equation in spherical coordinates.)

*Solution.* By plugging the spherical coordinate formulas for  $x, y, z$  in terms of  $\rho, \phi, \theta$  into the equation for the sphere, one gets the equation

$$\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \cos^2 \phi - 2\rho \cos \phi + 1 = 1$$

which simplifies to  $\rho = 2 \cos \phi$ . The cone  $z = \sqrt{x^2 + y^2}$  becomes just  $\phi = \pi/4$  in spherical coordinates. The integral is then

$$\begin{aligned} \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2 \cos \phi} (\rho^2) \rho^2 \sin \phi d\rho d\phi d\theta &= \int_0^{2\pi} d\theta \int_0^{\pi/4} \left[ \frac{\rho^5 \sin \phi}{5} \Big|_0^{2 \cos \phi} \right] d\phi \\ 2\pi \int_0^{\pi/4} \frac{32}{5} \cos^5 \phi \sin \phi d\phi &= \frac{64\pi}{5} \left( \frac{-\cos^6 \phi}{6} \right) \Big|_0^{\pi/4} = \frac{64\pi}{30} \left( 1 - \frac{1}{8} \right) = \frac{28\pi}{15} \end{aligned}$$