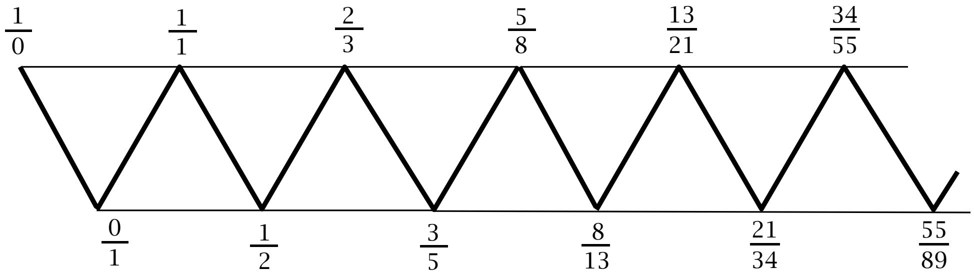
Infinite Continued Fractions

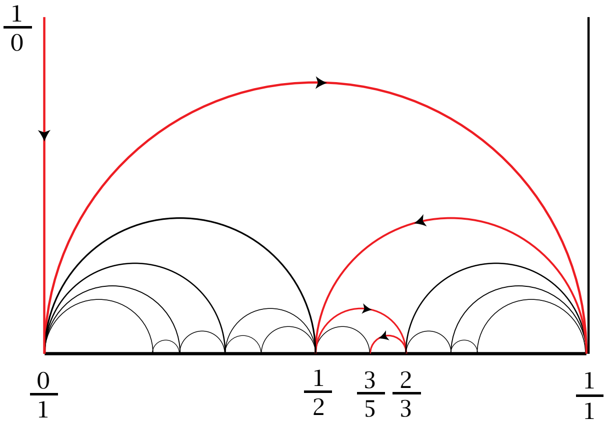
We know that all rational numbers can be represented as continued fractions $m + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_n}$, but what about irrational numbers? It turns out that these can be represented as *infinite* continued fractions $m + \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots$. A simple example is when all the a_i's are 1, the infinite continued fraction $\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots$, or in its expanded form:

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}$$

The corresponding strip of triangles is infinite:



Notice that these fractions are just the successive ratios of the famous Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, \cdots$ where each number is the sum of its two predecessors. The sequence of convergents is thus $0/1, 1/1, 1/2, 2/3, 3/5, 5/8, 8/13, \cdots$, the vertices along the zigzag path. Here is what the beginning of this zigzag path looks like in the Farey diagram:



What happens when we follow this path farther and farther? The path consists of an infinite sequence of semicircles, each one shorter than the preceding one and sharing a common endpoint. The left endpoints of the semicircles form an increasing sequence of numbers which have to be approaching a certain limiting value x. We know x has to be finite since it is certainly less than each of the right-hand endpoints of the semicircles, the convergents $1/1, 2/3, 5/8, \cdots$. Similarly the right endpoints of the semicircles form a decreasing sequence of numbers approaching a limiting value y greater than each of the left-hand endpoints $0/1, 1/2, 3/5, \cdots$. Obviously $x \le y$. Is it possible that x is not equal to y? If this happened, the infinite sequence of semicircles would be approaching the semicircle from x to y. Above this semicircle there would then be an infinite number of semicircles, all the semicircles in the infinite sequence. Between x and y there would have to be a rational numbers p/q (between any two real numbers there is always a rational number), so above this rational number there would be an infinite number of semicircles, hence an infinite number of triangles in the Farey diagram. But we know that there are only finitely many triangles above any rational number p/q, the triangles that appear in the strip for the continued fraction for p/q. This contradiction shows that x has to be equal to y. Thus the sequence of convergents along the edges of the infinite strip of triangles converges to a unique real number x. (This is why the convergents are called convergents, of course.)

There is a simple method for computing the actual value of x. We have

$$x = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\cdots$$

so if we take the reciprocals of both sides of this equation we get

$$\frac{1}{x} = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\cdots$$

The right side of this equation is just $1 + x$, so we can easily solve for x:

$$\begin{aligned} \frac{1}{x} &= 1 + x \\ 1 &= x + x^2 \\ x^2 + x - 1 &= 0 \\ x &= \frac{-1 \pm \sqrt{5}}{2} \end{aligned}$$

We know x is positive, so this rules out the negative root and we are left with the final value $x = (-1 + \sqrt{5})/2$. This number, approximately $.618$, goes by the name of the golden ratio. Some people seem to think it has an almost mystical significance — just do a google search for “phi” (the Greek symbol ϕ is used for this number by its fans).

The argument we gave that the infinite continued fraction $\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots$ has a unique real number value given by the limits of its sequence of convergents applies more generally to any infinite continued fraction. $m+\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots$. One just looks at the corresponding infinite strip of triangles in the Farey diagram, and by the same reasoning as before it follows that the left endpoints of the semicircles in the zigzag path converge to the same limit as the right endpoints of these semicircles.

It is also true that every irrational number x has an expression as an infinite continued fraction. In the Farey diagram the vertical line going upward from x passes through an infinite sequence of triangles, and these triangles form an infinite strip which corresponds to an infinite continued fraction.

The procedure for computing the infinite continued fraction $m+\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots$ for a given irrational number x is the same as for rational numbers, but it doesn't terminate after a finite number of steps. Recall the original example that we did:

$$\frac{67}{24} = 2+\frac{19}{24} = 2+\frac{1}{24/19} = 2+\frac{1}{1+5/19} = 2+\cfrac{1}{1+\cfrac{1}{19/5}}$$

$$= 2+\cfrac{1}{1+\cfrac{1}{3+4/5}} = 2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{5/4}}} = 2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{4}}}}$$

The sequence of steps is the following:

(1) Write $x=m+r_1$ where m is an integer and $0\le r_1<1$

(2) Write $1/r_1=a_1+r_2$ where a_1 is an integer and $0\le r_2<1$

(3) Write $1/r_2=a_2+r_3$ where a_2 is an integer and $0\le r_3<1$

and so on, repeatedly. Thus one first finds the largest integer $m\le x$, with r_1 the 'remainder', then one inverts r_1 and finds the greatest integer $a_1\le 1/r_1$, with r_2 the remainder, etc.

Here is how this works for $x=\sqrt{2}$:

(1) $\sqrt{2}=1+(\sqrt{2}-1)$ where $a_1=1$ since $\sqrt{2}$ is between 1 and 2. Before going on to step (2) we have to compute $\frac{1}{r_1}=\frac{1}{\sqrt{2}-1}$. Multiplying numerator and denominator by $\sqrt{2}+1$ gives $\frac{1}{\sqrt{2}-1}=\frac{1}{\sqrt{2}-1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1}=\sqrt{2}+1$. This is the number we use in the next step.

(2) $\sqrt{2}+1=2+(\sqrt{2}-1)$ since $\sqrt{2}+1$ is between 2 and 3.

Notice that something unexpected has happened: The remainder $r_2 = \sqrt{2}-1$ is exactly the same as the previous remainder r_1. There is then no need to do the calculation of $\frac{1}{r_2} = \frac{1}{\sqrt{2}-1}$ since we know it will have to be $\sqrt{2}+1$. This means that the next step (3) will be exactly the same as step (2), and the same will be true for all subsequent steps. Hence we get the continued fraction

$$\sqrt{2} = 1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\cdots$$

We can check this calculation by finding the value of the continued fraction in the same way that we did earlier for $\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots$. First we set $x = \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\cdots$. Taking reciprocals gives $1/x = 2 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\cdots = 2 + x$. This leads to the quadratic equation $x^2 + 2x - 1 = 0$, which has roots $x = -1 \pm \sqrt{2}$. Since x is positive we can discard the negative root. Thus we have $-1 + \sqrt{2} = \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\cdots$. Adding 1 to both sides of this equation gives the earlier formula for $\sqrt{2}$ as a continued fraction.

If we try the same procedure for $\sqrt{3}$ something slightly different happens:

(1) $\sqrt{3} = 1 + (\sqrt{3}-1)$ since $\sqrt{3}$ is between 1 and 2. Computing $\frac{1}{\sqrt{3}-1}$, we have $\frac{1}{\sqrt{3}-1} = \frac{1}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{\sqrt{3}+1}{2}$.

(2) $\frac{\sqrt{3}+1}{2} = 1 + (\frac{\sqrt{3}-1}{2})$ since $\sqrt{3}+1$ is between 2 and 3. Now we have a remainder $r_2 = \frac{\sqrt{3}-1}{2}$ which is different from the previous remainder $r_1 = \sqrt{3}-1$, so we have to compute $\frac{1}{r_2} = \frac{2}{\sqrt{3}-1}$, namely $\frac{2}{\sqrt{3}-1} = \frac{2}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1} = \sqrt{3}+1$.

(3) $\sqrt{3}+1 = 2 + (\sqrt{3}-1)$ since $\sqrt{3}+1$ is between 2 and 3.

Now this remainder $r_3 = \sqrt{3}-1$ is the same as r_1, so instead of the same step being repeated infinitely often, as happened for $\sqrt{2}$, the same two steps will repeat infinitely often. This means we get the continued fraction

$$\sqrt{3} = 1 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots$$

Checking this takes a little more work than before. We begin by isolating the part of the continued fraction that repeats periodically, so we set

$$x = \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots$$

Taking reciprocals, we get

$$\frac{1}{x} = 1 + \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots$$

Subtracting 1 from both sides gives

$$\frac{1}{x} - 1 = \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots$$

The next step will be to take reciprocals of both sides, so before doing this we rewrite the left side as $\frac{1-x}{x}$. Then taking reciprocals gives

$$\frac{x}{1-x} = 2 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots$$

Hence

$$\frac{x}{1-x} - 2 = \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots = x$$

Now we have the equation $\frac{x}{1-x} - 2 = x$ which can be simplified to the quadratic equation $x^2 + 2x - 2 = 0$, with roots $x = -1 \pm \sqrt{3}$. Again the negative root is discarded, and we get $x = -1 + \sqrt{3}$. Thus $\sqrt{3} = 1 + x = 1 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots$.

It is true in general that for every positive integer n the continued fraction for $\sqrt{n}$ has the form $m + \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots$ where the numbers a_i repeat periodically. The length of the period can be large. For example

$$\sqrt{46} = 6 + \overline{\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{6+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{12+}}$$

where the line over all the block of 12 terms indicates that this block is repeated infinitely often. This example illustrates two other curious facts about the continued fraction for a number $\sqrt{n}$:

(i) The last term of the period (12 in the example) is always twice the integer m (the initial 6).

(ii) If the last term of the period is omitted, the preceding terms form a palindrome, reading the same backwards as forwards.

It is natural to ask exactly which irrational numbers have continued fractions that are periodic, or at least *eventually* periodic, like for example

$$\frac{1}{2+}\,\frac{1}{4+}\,\overline{\frac{1}{5+}\,\frac{1}{7+}\,\frac{1}{3+}} = \frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{5+}\,\frac{1}{7+}\,\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{7+}\,\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{7+}\,\frac{1}{3+}\cdots$$

The answer is that the numbers whose continued fractions are eventually periodic are exactly the *quadratic irrationals*, the irrational numbers that satisfy a quadratic equation whose coefficients are integers. More concretely, quadratic irrationals are the numbers of the form $a + b\sqrt{n}$ where a and b are rational numbers and n is a positive integer that is not a perfect square.

What about irrational numbers that are not quadratic, such as $\sqrt[3]{2}$, π, or e, the base for natural logarithms? It just so happens that e has a continued fraction whose terms have a very nice pattern, even though they are not periodic or eventually periodic:

$$e = 2 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{8+}\,\frac{1}{1+}\cdots$$

However, for $\sqrt[3]{2}$ and π the continued fractions have no known pattern. For π the continued fraction begins

$$\pi = 3 + \frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{292+}\,\cdots$$

Here the first four convergents are 3, $22/7$, $322/105$, and $355/113$. We recognize $22/7$ as the familiar approximation $3\frac{1}{7}$ to π. The convergent $355/113$ is a particularly good approximation to π since its decimal expansion begins 3.14159282 whereas $\pi = 3.1415926535\cdots$. It is no accident that the convergent $355/113$ obtained by truncating the continued fraction just before the 292 term gives a good approximation to π since it is a general fact that a convergent immediately preceding a large term in a continued fraction always gives an especially good approximation.