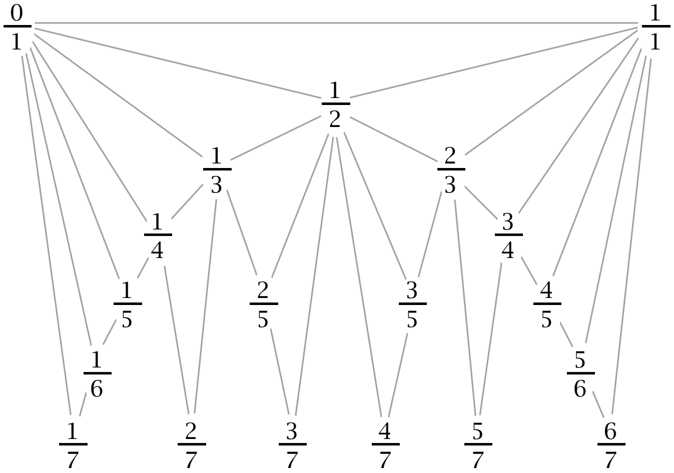
Farey Series

We can build the set of rational numbers by starting with the integers and then inserting in succession all the halves, thirds, fourths, fifths, and so on. Let us look at what happens if we restrict to rational numbers between 0 and 1. Starting with 0 and 1 we first insert $1/2$, then $1/3$ and $2/3$, then $1/4$ and $3/4$, skipping $2/4$ which we already have, then inserting $1/5$, $2/5$, $3/5$, and $4/5$, then $1/6$ and $5/6$, etc. This process can be pictured as in the following diagram:

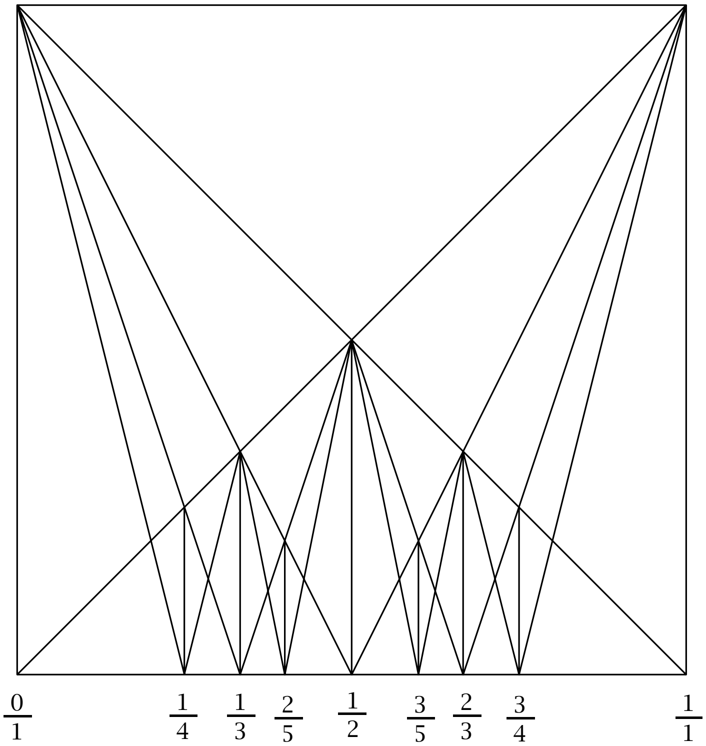


The interesting thing to notice is that each time a new number is inserted, it forms the third vertex of a triangle whose other two vertices are its two nearest neighbors among the numbers already listed, and if these two neighbors are a/b and c/d then the new vertex is exactly the mediant $\frac{a+c}{b+d}$. This curious phenomenon seems to have been first observed by a geologist and amateur mathematician named Farey in the early 1800s. In his honor, the sequence of fractions a/b between 0 and 1 with denominator less than or equal to a given number n is called the nth *Farey series* F_n. For example, here is F_7:

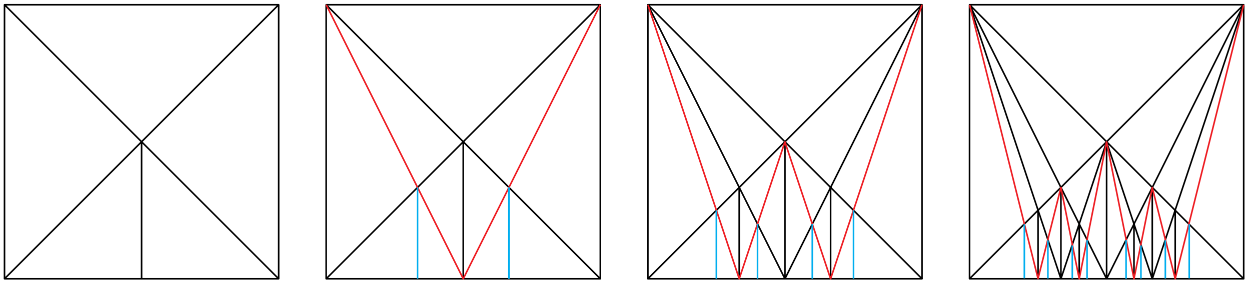
$$\frac{0}{1},\ \frac{1}{7},\ \frac{1}{6},\ \frac{1}{5},\ \frac{1}{4},\ \frac{2}{7},\ \frac{1}{3},\ \frac{2}{5},\ \frac{3}{7},\ \frac{1}{2},\ \frac{4}{7},\ \frac{3}{5},\ \frac{2}{3},\ \frac{5}{7},\ \frac{3}{4},\ \frac{4}{5},\ \frac{5}{6},\ \frac{6}{7},\ \frac{1}{1}$$

These numbers trace out the up-and-down path across the bottom of the figure above. For the next Farey series F_8 we would insert $1/8$ between $0/1$ and $1/7$, $3/8$ between $1/3$ and $2/5$, $5/8$ between $3/5$ and $2/3$, and finally $7/8$ between $6/7$ and $1/1$.

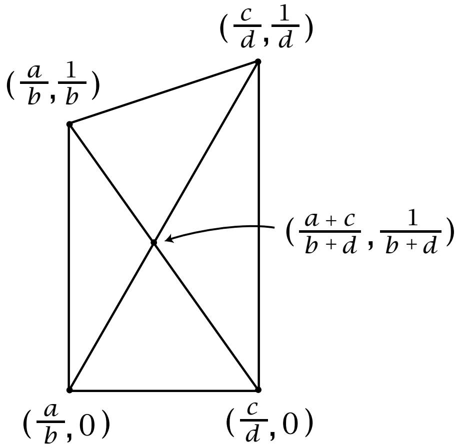
A cleaner way to draw this diagram is shown in the next figure.



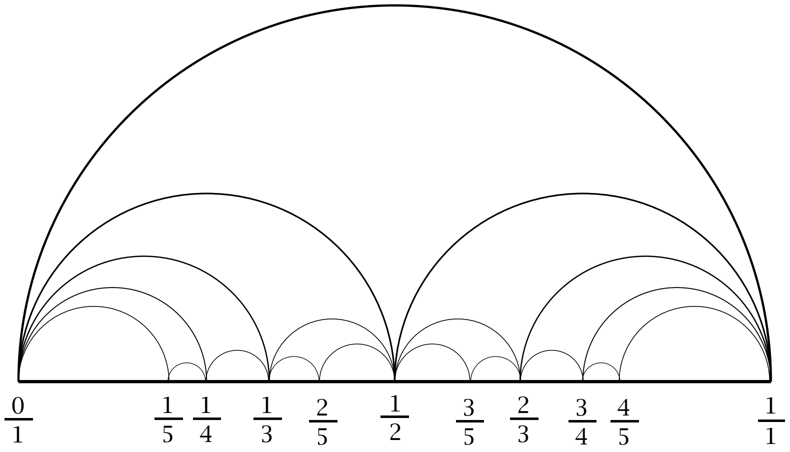
One can construct this diagram in stages. Start with a square together with its diagonals and a vertical line from their intersection point down to the bottom edge of the square. Next, connect the resulting midpoint of the lower edge of the square to the two upper corners of the square and drop vertical lines down from the two new intersection points this produces. Now add a W-shaped zigzag and drop verticals again. It should then be clear how to continue.



A nice feature of this construction is that if we start with a square whose sides have length 1 and place this square so that its bottom edge lies along the x-axis with the lower left corner of the square at the origin, then the construction assigns labels to the vertices along the bottom edge of the square that are exactly the x coordinates of these points. Thus the vertex labeled $1/2$ really is at the midpoint of the bottom edge of the square, and the vertices labeled $1/3$ and $2/3$ really are $1/3$ and $2/3$ of the way along this edge, and so forth. In order verify this fact the key observation is the following: For a vertical line segment in the diagram whose lower endpoint is at the point $(a/b, 0)$ on the x-axis, the upper endpoint is at the point $(a/b, 1/b)$. This is obviously true at the first stage of the construction, and it continues to hold at each successive stage since for a quadrilateral whose four vertices have coordinates as shown in the figure at the right, the two diagonals intersect at the point $(\frac{a+c}{b+d}, \frac{1}{b+d})$. This can be verified by writing down equations for the two diagonals and checking that the point $(x, y) = (\frac{a+c}{b+d}, \frac{1}{b+d})$ satisfies both equations.

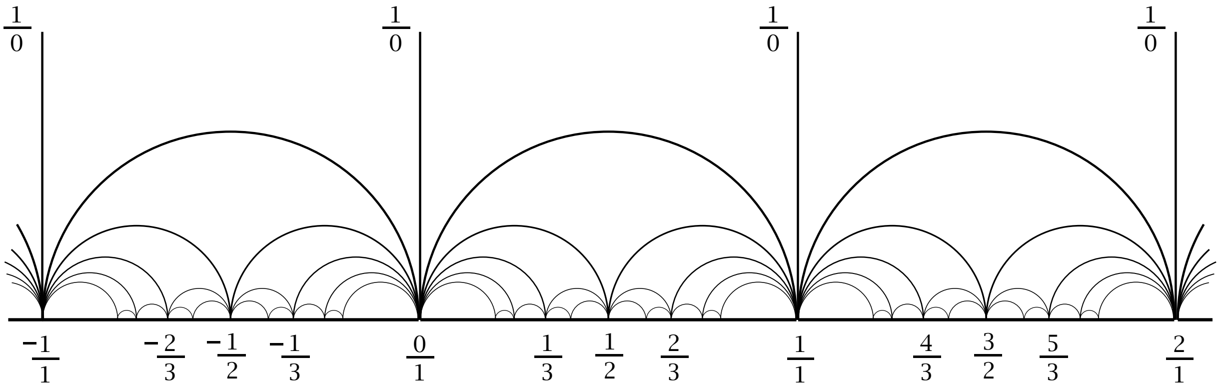


Going back to the diagram consisting of the square with the line segments in it, the most important thing in this diagram for our purposes is the triangles, not the vertical lines. We can get rid of all the vertical lines by shrinking each one to its lower endpoint, converting each triangle into a curvilinear triangle, as shown in the following variant of the diagram:



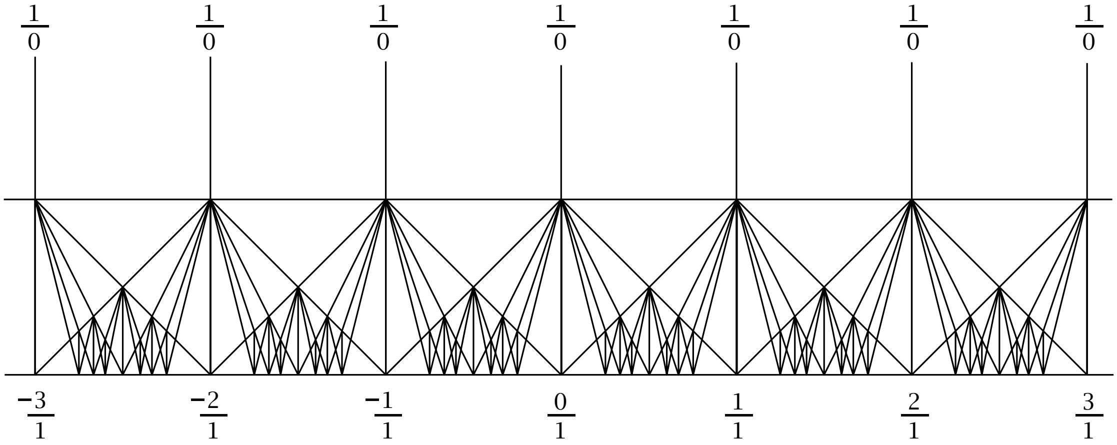
This looks more like a portion of the diagram we started with at the beginning of the chapter, the main difference being that the shapes of the curvilinear triangles have been somewhat distorted. The advantage of the present diagram is that the labels on the vertices are exactly in their correct places along the x-axis, so the vertex labeled a/b is exactly at the point a/b on the x-axis.

This diagram can be enlarged so as to include similar diagrams for fractions between all pairs of adjacent integers, not just 0 and 1, all along the x-axis:



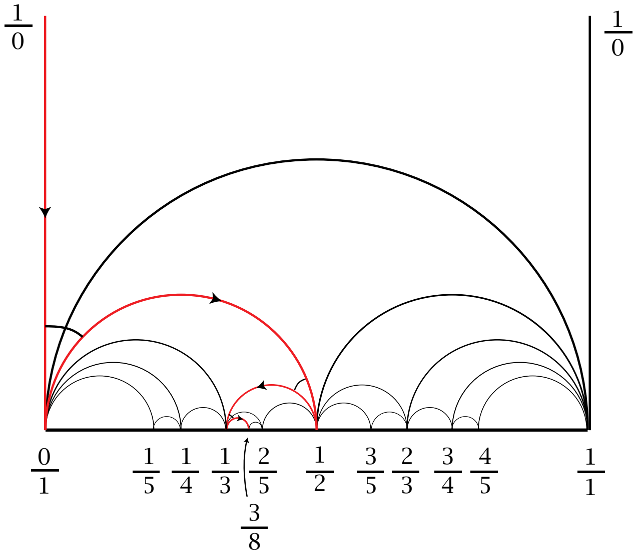
We can also put in vertical lines at the integer points, extending upward to infinity. These correspond to the edges having one endpoint at the vertex $1/0$ in the original circular diagram.

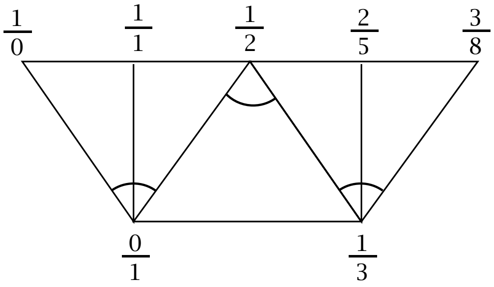
We should give names to these diagrams for more convenient reference. Let us call the diagram we have just drawn the *standard Farey diagram* and the one at the beginning of the chapter the *circular Farey diagram*. We could also form a Farey diagram from copies of the square:



We will call this the *linear Farey diagram*.

In the standard Farey diagram the zigzag paths that pass through vertices given by the successive convergents for a continued fraction $\frac{p}{q} = m + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_n}$ become 'pinball paths', starting down the vertical line from $1/0$ to $m/1$, then turning left across a_1 triangles, then right across a_2 triangles, then left across a_3 triangles, continuing to alternate left and right turns until reaching the final vertex p/q. One can see from this that the convergents are alternately smaller than and greater than p/q. Also, the triangles that form the strip of triangles for p/q are exactly the triangles that lie directly above the point p/q on the x-axis. Here is the simple example of $p/q = 3/8$:





Note that the five triangles of the strip correspond to the four curvilinear triangles lying directly above $3/8$ in the diagram at the left, plus the fifth 'triangle' extending upward to infinity, bounded on the left and right by the vertical lines above $0/1$ and $1/1$, and bounded below by the semicircle from $0/1$ to $1/1$.