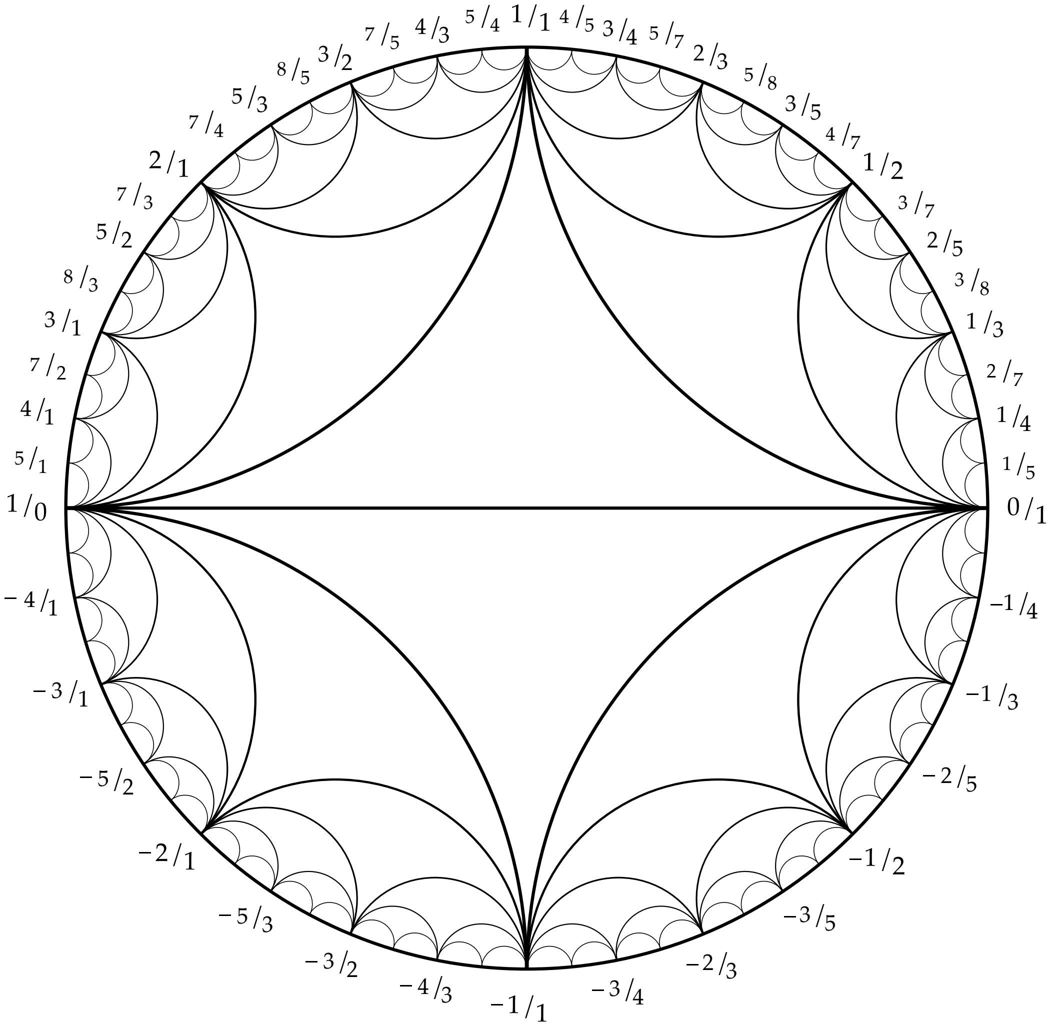
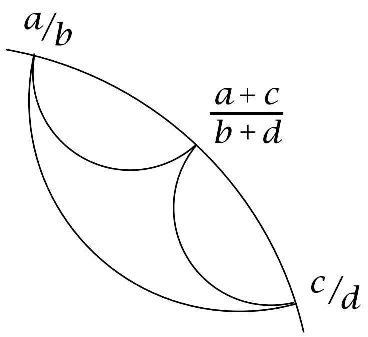
One of our main goals is to relate numbers of various kinds to geometry. The simplest sorts of numbers are integers, along with their ratios, the rational numbers. There is a very interesting diagram, not as well known as it ought to be, displaying rational numbers and certain relations between them that we will be exploring:



What is shown here is not the whole diagram but only a finite part of it. The actual diagram has infinitely many curvilinear triangles, getting smaller and smaller out near the boundary circle. The diagram can be constructed by first inscribing the two big triangles in the circle, then adding the four triangles that share an edge with the two big triangles, then the eight triangles sharing an edge with these four, then sixteen more triangles, and so on forever.

The vertices of all the triangles are labeled with fractions a/b, including the fraction $1/0$ for ∞, according to the following scheme. In the upper half of the diagram first label the vertices of the big triangles $0/1$, $1/1$, and $1/0$ as shown. Then by induction, if the labels at the two ends of the long edge of a triangle are a/b and c/d, the label on the third vertex of the triangle is $\frac{a+c}{b+d}$. This fraction is called the *mediant* of a/b and c/d. (If only we had been taught this simplified form of fraction addition in elementary school!)



The labels in the lower half of the diagram follow the same scheme, starting with the labels $0/1$, $-1/1$, and $-1/0$ on the large triangle. Using $-1/0$ instead of $1/0$ as the label of the vertex at the far left means that we are regarding $+\infty$ and $-\infty$ as the same.

The labels in the lower half of the diagram are the negatives of those in the upper half, and the labels in the left half are the reciprocals of those in the right half. Here are three other properties of the labeling:

(i) The rational number labels occur in their proper order around the circle, increasing from $-\infty$ to $+\infty$ as one goes around the circle in the counterclockwise direction.

(ii) Every rational number occurs eventually as the label of a vertex.

(iii) There is an edge joining two vertices labeled a/b and c/d exactly when $ad-bc = \pm 1$.

Checking that the rational numbers occur in their proper order is easy. It suffices to look at the upper half of the diagram where all numbers are positive. What we want to show is that the mediant $\frac{a+c}{b+d}$ is always a number between $\frac{a}{b}$ and $\frac{c}{d}$. Thus we want to see that if $\frac{a}{b} > \frac{c}{d}$ then $\frac{a}{b} > \frac{a+c}{b+d} > \frac{c}{d}$. Since we are dealing with positive numbers, the inequality $\frac{a}{b} > \frac{c}{d}$ is equivalent to $ad > bc$, and $\frac{a}{b} > \frac{a+c}{b+d}$ is equivalent to $ab + ad > ab + bc$, which follows from $ad > bc$. Similarly, $\frac{a+c}{b+d} > \frac{c}{d}$ is equivalent to $ad + cd > bc + cd$, which also follows from $ad > bc$.

To show that all rational numbers eventually occur as labels in the diagram will involve a new idea which is our next topic.

Continued Fractions

Here are two typical examples of continued fractions:

$$\frac{7}{16} = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2}}} \qquad\qquad \frac{67}{24} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}}}$$

To verify the equalities one starts in the lower right corner of the continued fraction and works one's way upward. For example in the continued fraction for $\frac{7}{16}$ one starts with $3 + \frac{1}{2} = \frac{7}{2}$, then taking 1 over this gives $\frac{2}{7}$, and adding the 2 to this gives $\frac{16}{7}$, and finally 1 over this gives $\frac{7}{16}$.

Here is the general form of a continued fraction:

$$\frac{p}{q} = m + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}$$

To write the continued fraction in more compact form on a single line one can write it as $m + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_n}$. For example $\frac{7}{16} = \frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2}$.

To compute the continued fraction for a given rational number one starts in the upper left corner and works one's way downward, as the following example shows:

$$\frac{67}{24} = 2 + \frac{19}{24} = 2 + \frac{1}{24/19} = 2 + \frac{1}{1 + 5/19} = 2 + \cfrac{1}{1 + \cfrac{1}{19/5}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{3 + 4/5}} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{5/4}}} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}}}$$

This process is known as the *Euclidean Algorithm*. It consists of repeated division, at each stage dividing the previous remainder into the previous divisor. The procedure for $67/24$ is shown at the right. Note that the numbers in the shaded box are the numbers that form the continued fraction. These are the quotients of the successive divisions. They are sometimes called the *partial quotients* of the original fraction.

$$\begin{array}{rcl}
67 & = & \boxed{2}\cdot 24 + 19 \\
24 & = & \boxed{1}\cdot 19 + 5 \\
19 & = & \boxed{3}\cdot 5 + 4 \\
5 & = & \boxed{1}\cdot 4 + 1 \\
4 & = & \boxed{4}\cdot 1 + 0
\end{array}$$

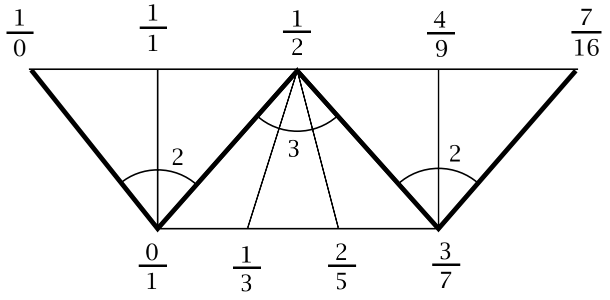
One of the classical uses for the Euclidean algorithm is to find the greatest common divisor of two given numbers. If one applies the algorithm to two numbers p and q, dividing the smaller into the larger, then the remainder into the first divisor, and so on, then the greatest common divisor of p and q turns out to be the last nonzero remainder. For example, starting with $p = 72$ and $q = 201$ the calculation is shown at the right, and the last nonzero remainder is 3, which is the greatest common divisor of 72 and 201. (In fact the fraction $201/72$ equals $67/24$, which explains why the successive quotients for this example are the same as in the preceding example.) It is easy to see from the displayed equations why 3 has to be the greatest common divisor of 72 and 201, since from the first equation it follows that any divisor of 72 and 201 must also divide 57, then the second equation shows it must divide 15, and the fourth equation shows it must divide 3, the last nonzero remainder. Conversely, if a number divides the last nonzero remainder 3 then the last equation shows it must also divide the 12, and the next-to-last equation then shows it must divide 15, and so on until we conclude that it divides all the numbers not in the shaded rectangle, including the original two numbers 72 and 201. The same reasoning applies in general.

$$\begin{array}{rcl}
201 &=& 2 \cdot 72 + 57 \\
72 &=& 1 \cdot 57 + 15 \\
57 &=& 3 \cdot 15 + 12 \\
15 &=& 1 \cdot 12 + \textcircled{3} \\
12 &=& 4 \cdot 3 + 0
\end{array}$$

A more obvious way to try to compute the greatest common divisor of two numbers would be to factor each of them into a product of primes, then look to see which primes occurred as factors of both, and to what power. But to factor a large number into its prime factors is a very laborious and time-consuming process. For example, even a large computer would have a hard time factoring a number of a hundred digits into primes, so it would not be feasible to find the greatest common divisor of a pair of hundred-digit numbers this way. However, the computer would have no trouble at all applying the Euclidean algorithm to find their greatest common divisor.

Having seen what continued fractions are, let us now see what they have to do with the large diagram at the beginning of this section. Some examples will illustrate this best, so let us first look at the continued fraction for $7/16$ again. This has $2, 3, 2$ as its sequence of partial quotients. We use these three numbers to build a strip of three large triangles subdivided into 2, 3, and 2 smaller triangles, from left to right:

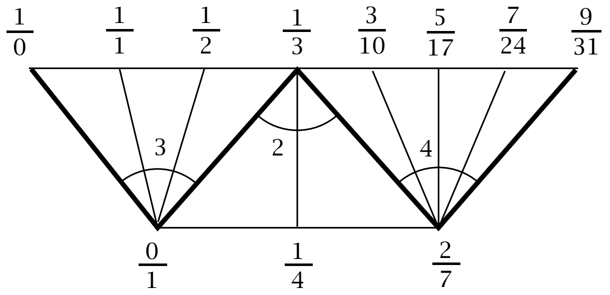
$$\frac{7}{16} = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2}}}$$



We can think of the diagram as being formed from three “fans”, where the first fan is made from the first 2 small triangles, the second fan from the next 3 small triangles, and the third fan from the last 2 small triangles. Now we begin labeling the vertices of this strip. On the left edge we start with the labels $1/0$ and $0/1$. Then we use the mediant rule for computing the third label of each triangle in succession as we move from left to right in the strip. Thus we insert, in order, the labels $1/1$, $1/2$, $1/3$, $2/5$, $3/7$, $4/9$, and finally $7/16$.

Was it just an accident that the final label was the fraction $7/16$ that we started with, or does this always happen? Doing more examples should help us decide. Here is a second example:

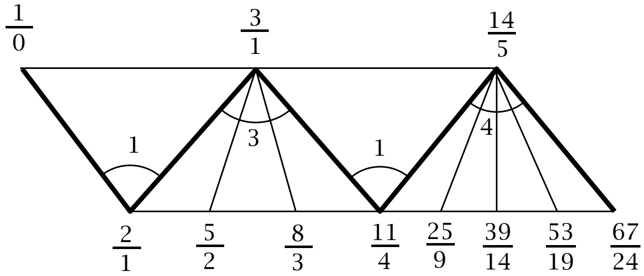
$$\frac{9}{31} = \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{4}}}$$



Again the final vertex on the right has the same label as the fraction we started with. The reader is encouraged to try more examples to make sure we are not rigging things to get a favorable outcome by only choosing examples that work.

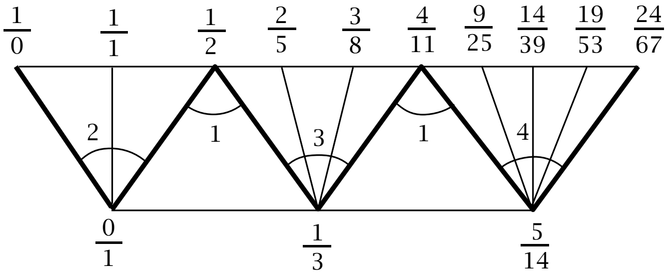
In fact this always works for fractions p/q between 0 and 1. For fractions larger than 1 the procedure works if we modify it by replacing the label $0/1$ with the initial integer m in the continued fraction $m + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_n}$. This is illustrated by the $67/24$ example:

$$\frac{67}{24} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}}}$$



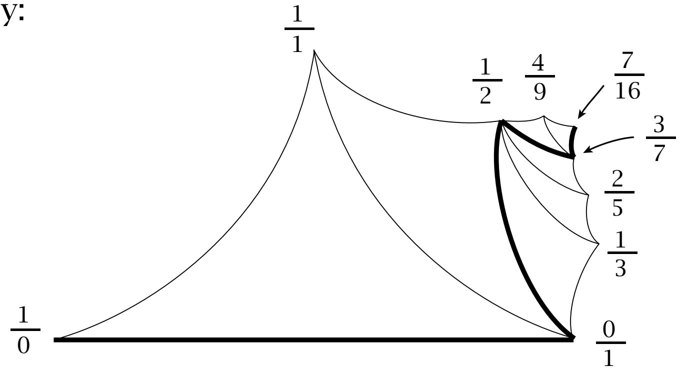
For comparison, here is the corresponding strip for the reciprocal, $24/67$:

$$\frac{24}{67} = \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}}}}$$

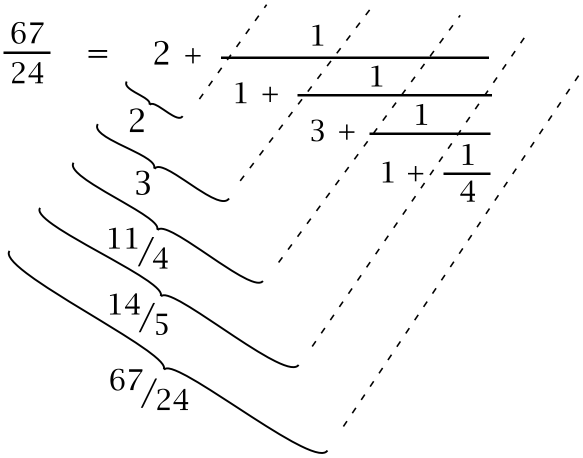


Now at last we can return to the large circular diagram. Since the rule for labeling vertices in the triangles along the horizontal strip for a fraction p/q is the mediant rule, each of the triangles in the strip is a triangle in the circular diagram, somewhat distorted in shape, and the strip of triangles can be regarded as a sequence of adjacent triangles in the diagram. Here is what this looks like for the fraction $7/16$, slightly distorted for the sake of visual clarity:

$$\frac{7}{16} = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2}}}$$



In each of the horizontal strips of triangles there is a zigzag path indicated by the heavily shaded edges, starting at $1/0$ on the left and ending at the final vertex p/q on the right. The vertices that this zigzag path passes through have a special significance. They are the fractions that occur as the values of successively larger initial portions of the continued fraction, as illustrated in the following example:



These fractions are called the *convergents* for the given fraction. Thus the convergents for $67/24$ are 2, 3, $11/4$, $14/5$, and $67/24$ itself.

SUMMARY: *The convergents for the continued fraction*

$$\frac{p}{q} = m + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}$$

are the vertices along a zigzag path in the big diagram formed by a sequence of edges starting at $1/0$ *and ending at* p/q*. The path starts along the edge from* $1/0$ *to* $m/1$*, then turns left across a fan of* a_1 *triangles, then right across a fan of* a_2 *triangles, etc., finally ending at* p/q*.*

In particular, since every positive rational number has a continued fraction expansion, we see that every positive rational number occurs eventually as the label of some vertex in the upper half of the diagram. All negative rational numbers then occur as labels in the lower half.