Notes on Lie Groups Acting on Symplectic Manifolds

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1 Haar Measure

Definition 1.1. A topological group is a group $G$ that is also a topological space, such that the maps $G \times G \to G, \quad (x, y) \mapsto xy$ and $G \to G, \quad x \mapsto x^{-1}$ are continuous. For each $a \in G$, we obtain two continuous homomorphisms $L_a: G \to G$ and $R_a: G \to G$, called left and right multiplication, respectively, defined by $L_a(g) = ag$ and $R_a(g) = ga$ for all $a \in G$. 

Suppose $G$ is locally compact. Then the $\sigma$-algebra generated by the compact subsets of $G$ is called the **Borel algebra** of $G$, denoted by $\mathcal{B}$. A **Borel set** is an element of $\mathcal{B}$. The set $\mathcal{B}$ is invariant under the maps $L_a$ and $R_a$, for each $a \in G$.

A measure $\gamma$ on $\mathcal{B}$ is called **left invariant**, (respectively **right invariant**), if
\[
\gamma(L_a(S)) = \gamma(S), \left(\text{respectively}, \quad \gamma(R_a(S)) = \gamma(S)\right)
\]
for all $a \in G$ and $S \in \mathcal{B}$. The measure $\gamma$ is **regular** if

1. $\gamma(K) < \infty$ for every compact $K \subset G$;
2. every $E \in \mathcal{B}$ is **outer regular**:
   \[
   \gamma(E) = \inf\{\gamma(U) \mid E \subset U \text{ and } U \in \mathcal{B} \text{ is open}\};
   \]
3. every $E \in \mathcal{B}$ is **inner regular**:
   \[
   \gamma(E) = \sup\{\gamma(K) \mid K \subset E \text{ and } K \text{ is compact}\}.
   \]

**Theorem 1.2.** Up to a positive multiplicative constant, there exists a unique left, (respectively right) invariant measure $\gamma$ on $\mathcal{B}$ that is countably additive and regular, and such that $\gamma(U) > 0$ for every open non-empty $U \in \mathcal{B}$.

**Remark 1.3.** The measure described in Theorem 1.2 is called a **Haar measure**, introduced by and named after Alfréd Haar, a Hungarian mathematician, in about 1932.

**Lemma 1.4.** For every subset $S \subset G$, let $S = \{g^{-1} \mid g \in S\}$. Note that $\mathcal{B}$ is closed under this set transformation. Let $\gamma$ be a Haar measure on $\mathcal{B}$, and define the measure $\gamma^{-1}$ by
\[
\gamma^{-1}(S) = \gamma(S^{-1}).
\]
Then $\gamma^{-1}$ is a right invariant measure.

**Proof.** Let $a \in G$ and $S \in \mathcal{B}$. Observe that $R_a(S) = (L_a^{-1}(S^{-1}))$. Hence
\[
\gamma^{-1}(R_a(S)) = \gamma(L_a^{-1}(S^{-1})) = \gamma(S^{-1}) = \gamma^{-1}(S).
\]
QED
Remark 1.5. Let $\gamma$ be a measure on a Lie group $G$, and let $f: G \to \mathbb{R}^n$ be a smooth function with finite integral over $G$. Saying that $\gamma$ is left invariant is equivalent to saying that for all $h \in G$, we have

$$\int_G f(h \cdot g) \, d\gamma(g) = \int_G f(g) \, d\gamma(x).$$

Lemma 1.6. Let $G$ be a Lie group with measure $\gamma$, let $V$ and $W$ be finite-dimensional vector spaces, let $f: G \to V$ be a smooth function with finite integral over $G$, and let $L: V \to W$ be a linear transformation. Then

$$L\left[ \int_G f(g) \, d\gamma(g) \right] = \int_G L[f(g)] \, d\gamma(g).$$

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis for $V$. Then there exist smooth functions $a_1, \ldots, a_n: G \to \mathbb{R}$ such that for each $g \in G$ we have

$$f(g) = \sum_{i=1}^n a_i(g) v_i.$$

Then

$$L\left[ \int_G f(g) \, d\gamma(g) \right] = L\left[ \int_G \sum_{i=1}^n a_i(g) v_i \, d\gamma(g) \right] = L\left[ \sum_{i=1}^n \left( \int_G a_i(g) \, d\gamma(g) \right) v_i \right]$$

$$= \sum_{i=1}^n \left( \int_G a_i(g) \, d\gamma(g) \right) L(v_i) = \int_G \sum_{i=1}^n a_i(g) L(v_i) \, d\gamma(g)$$

$$= \int_G L[f(g)] \, d\gamma(g).$$

QED

2 A Little Bit About Lie Groups

Let $G$ be a Lie group. For each $g \in G$, let $L_g: G \to G$ be the smooth function given by $L_g(a) = ga$ for all $a \in G$. Note that $L_g$ is a diffeomorphism, with inverse $L_{g^{-1}}$. These
diffeomorphisms allow us to formulate a canonical identification of the tangent spaces of $G$. Specifically, we can identify each tangent space $T_g G$ with the Lie algebra $\mathfrak{g} := T_e G$, where $e \in G$ is the identity of $G$, via the linear isomorphism

$$T_e L_g : \mathfrak{g} \to T_g G.$$ 

Every finite dimensional real vector space $V$ is a Lie group under vector addition. Its Lie algebra is the tangent space at the origin, $T_0 V$, so every tangent space $T_v V$ can be canonically identified with $T_0 V$. We can go one step further and identify the vector space $V$ with its Lie algebra $T_0 V$. For each $v \in V$, let $c_v : \mathbb{R} \to V$ be the curve given by $t \mapsto t \cdot v$. Define $\Gamma : V \to T_0 V$ by

$$\Gamma(v) = \dot{c}_v(0) \in T_0 V$$

for each $v \in V$. Note that $\Gamma$ is linear and fairly obvious must have trivial kernel, so it is a linear isomorphism. Thus every tangent space $T_v V$ of $V$ can be canonically identified with $V$. We will be canonically identifying vector spaces all over the place, and this is the justification (hopefully).

Observe that these canonical identifications rely on the velocity vector field of a smooth curve. Defining this requires the use of the standard vector field $\partial/\partial t$ on $\mathbb{R}$. This is really the only non-obvious structure whose existence we have to assume in order to construct all of these identifications.

All of this is (probably) equivalent to assuming always that we are working with $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$ with its standard coordinates, and hence its standard vector fields. Of course, for $\mathbb{R}$, this standard vector field is precisely $\partial/\partial t$.

The identification of every tangent space of $\mathbb{R}$ with itself is precisely what allows us to construct a differentiable 1-form $df$ from each function $f \in C^\infty(\mathcal{M})$. For each $r \in \mathbb{R}$, let $\Gamma_r : \mathbb{R} \to T_r \mathbb{R}$ denote the canonical linear isomorphism. Then for each $x \in \mathcal{M}$ and $v \in T_x \mathcal{M}$, we have

$$df(v) := \Gamma_{f(x)}^{-1} \circ T_x f(v).$$

**Definition 2.1.** The Lie group $G$ acts on itself, and also on its Lie algebra. The action of $G$ on itself is called the **conjugation action**, given by $g \in G \mapsto \Psi_g \in \text{Aut}(G)$, where $\Psi_g(h) = ghg^{-1}$ for all $h \in G$. The action of $G$ on $\mathfrak{g}$ is called the **adjoint action** of $G$. It
is denoted $\text{Ad}: G \to \mathfrak{S}_L(\mathfrak{g})$, and given by

$$g \in G \mapsto T_e \Psi_g \in \mathfrak{S}_L(\mathfrak{g}).$$

The Lie group $G$ also acts on $\mathfrak{g}^\ast$. This is called the coadjoint action of $G$. It is the dual of the action $\text{Ad}$, and is denoted $\text{Ad}^*: G \to \mathfrak{S}_L(\mathfrak{g}^\ast)$, and given by

$$\text{Ad}^*(g)(\alpha)(\xi) := \alpha(\text{Ad}(g^{-1})(\xi)) = \alpha(T_e \Psi_{g^{-1}}(\xi)).$$

$\triangle$

We have a smooth map $\text{Ad}: G \to \mathfrak{S}_L(\mathfrak{g})$. Taking the derivative at the identity $e \in G$ gives a smooth map $T_e \text{Ad}: \mathfrak{g} \to T_I(\mathfrak{S}_L(\mathfrak{g}))$, where $I$ denotes the identity map on $\mathfrak{g}$. Since $\mathfrak{S}_L(\mathfrak{g})$ is an open subset of the vector space $\mathfrak{gl}(\mathfrak{g})$, every tangent space of $\mathfrak{S}_L(\mathfrak{g})$ is canonically isomorphic to $\mathfrak{gl}(\mathfrak{g})$. Let $\Gamma_I: \mathfrak{gl}(\mathfrak{g}) \to T_I(\mathfrak{S}_L(\mathfrak{g}))$ be this isomorphism. Define $\text{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by

$$\text{ad} := \Gamma_I^{-1} \circ T_e \text{Ad}.$$ 

This allows us to define a bilinear bracket operation on $\mathfrak{g}$, given by

$$[X, Y] := \text{ad}(X)(Y) \in \mathfrak{g}.$$ 

This bracket actually satisfies the Jacobi identity, so it makes $\mathfrak{g}$ a Lie algebra in the abstract sense.

**Remark 2.2.** When $\mathfrak{g}$ is identified with the left invariant vector fields on $G$, the bracket on $\text{lie}$ given by $\text{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ corresponds to the standard Lie bracket of two vector fields. Hence $\mathfrak{g}$ and the left invariant vector fields on $G$ are isomorphic as Lie algebras. $\diamondsuit$

For each Lie group $G$, we have the **exponential map**, $\exp_G: \mathfrak{g} \to G$. We usually omit the subscript from $\exp$ if there is no confusion.

**Lemma 2.3.** Let $X \in \mathfrak{g}$, and let $c: \mathbb{R} \to G$ be the smooth curve given by $t \mapsto \exp(tX)$. Then $\dot{c}(0) = X$.

**Theorem 2.4.** Let $X \in \mathfrak{g}$, let $I$ denote the identity map on $\mathfrak{g}$, and let $\Gamma_I: \mathfrak{gl}(\mathfrak{g}) \to T_I(\mathfrak{S}_L(\mathfrak{g}))$ be the canonical linear isomorphism from $\mathfrak{gl}(\mathfrak{g})$ to the Lie algebra $T_I(\mathfrak{S}_L(\mathfrak{g}))$ of the Lie group $\mathfrak{S}_L(\mathfrak{g})$. Then

$$\text{Ad}(\exp_G X) = \exp_{\mathfrak{gl}(\mathfrak{g})} \left( \Gamma_I(\text{ad}X) \right).$$
Remark 2.5. The main formula of Theorem 2.4 is often written without distinguishing between the two exponential maps, and without explicitly writing the identification of $\mathfrak{gl}(\mathfrak{g})$ and $T_I(\mathfrak{g}L(\mathfrak{g}))$. This allows the much more ascetically pleasing formula

$$\text{Ad}(\exp X) = \exp(\text{ad}X).$$

With these notational relaxations, Theorem 2.4 is equivalent to requiring that the following diagram commutes.

$$
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\text{ad}} & \mathfrak{gl}(\mathfrak{g}) \\
\exp & \downarrow & \exp \\
G & \xrightarrow{\text{Ad}} & \mathfrak{gL}(\mathfrak{g}) \\
\end{array}
$$

Combining Theorem 2.4 and Lemma 2.3 yields the following result.

Proposition 2.6. Let $X, Y \in \mathfrak{g}$, and let $c: \mathbb{R} \to \text{lie}$ be the smooth curve given by $t \mapsto \text{Ad}((\exp(tX))(Y))$. Let $\Gamma_Y: \mathfrak{g} \to T_Y\mathfrak{g}$ be the canonical linear isomorphism. Then

$$\dot{c}(0) = \Gamma_Y([X,Y]).$$

3 Formulas for Forms and Fields

Let $X \in \mathfrak{X}(\mathcal{M})$. Then the Lie derivative $\mathcal{L}_X$ is a map from $k$-forms to $k$-forms, and the interior product $\iota(X)$ is a map from $k$-forms to $(k-1)$-forms. As usual, the exterior derivative $d$ is map from $(k-1)$-forms to $k$-forms. There are many connections among these three maps.

Theorem 3.1. Let $X,Y \in \mathfrak{X}(\mathcal{M})$. Then

(i) $\iota(X)\iota(Y) + \iota(Y)\iota(X) = 0$;

(ii) $\iota([X,Y]) = \mathcal{L}_X\iota(Y) - \iota(Y)\mathcal{L}_X$;

(iii) $\mathcal{L}_{[X,Y]} = \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X$, the generalized Jacobi identity;

(iv) $\mathcal{L}_X = d\iota(X) + \iota(X)d$, the homotopy identity, or Cartan’s identity;
(v) \( d\mathcal{L}_X - \mathcal{L}_X d = 0; \) and
(vi) \( d^2 = 0. \)

These identities can all be proved in local coordinates.

**Definition 3.2.** The zero section of the tangent bundle \( TM \) of \( M \) is the map \( z: M \to TM \) defined by \( x \mapsto 0_x \in T_x M. \)

**Remark 3.3.** The image of \( z(M) \) of the zero section of the tangent bundle \( TM \) of \( M \) is (canonically) diffeomorphic to \( M. \) In other words, the zero section \( z: M \to TM \) is a diffeomorphism onto its image. Therefore the differential of the zero section, at each \( x \in M, \) is a diffeomorphism \( z_x: T_x M \to T_{0_x}(TM). \)

Also, the zero section is a natural operator in the following sense. Suppose \( f: M \to N \) is a smooth map between manifolds, and \( z_M: M \to TM \) and \( z_N: N \to TN \) be zero sections. Then the following diagram commutes.

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow{z_M} & & \downarrow{z_N} \\
TM & \xrightarrow{f_*} & TN
\end{array}
\]

Using the chain rule, this implies that the differential of the zero section is also natural, so there is another commutative diagram, for each \( x \in M. \)

\[
\begin{array}{ccc}
T_x M & \xrightarrow{f_*} & T_{f(x)} N \\
\downarrow{(z_M)_*} & & \downarrow{(z_N)_{f(x)}} \\
T_{0_x}(TM) & \xrightarrow{(f_*)_0} & T_{0_{f(x)}}(TN)
\end{array}
\]

**Definition 3.4.** Let \( c: \mathbb{R} \to M \) be a smooth function. Then the velocity vector field of \( c \) is the vector field \( \dot{c}: \mathbb{R} \to Tc(\mathbb{R}) \subset TM \) defined by

\[
\dot{c}(t_0) = T_{c(t_0)}(\frac{\partial}{\partial t})|_{t=t_0} = (T_{t_0}c \circ \frac{\partial}{\partial t})(t_0).
\]
The velocity vector field is natural in the following sense.

**Lemma 3.5.** Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds, let $c : \mathbb{R} \to \mathcal{M}$ be a smooth curve, and let $\phi : \mathcal{M} \to \mathcal{N}$ be a smooth map. Then

$$(\phi \circ c)'(t) = T_{c(t)} \phi(c'(t))$$

for all $t \in \mathbb{R}$.

**Proof.** For each $t \in \mathbb{R}$, we have

$$(\phi \circ c)'(t) = \left[ T_{t} \phi \circ c \circ \frac{\partial}{\partial t} \right](t)$$

where $T_{t} \phi \circ c$ is in $T_{\phi \circ c(t)} \mathcal{N}$. By taking another derivative, we obtain a map $\phi'' : \mathbb{R} \to T(\mathcal{T} \mathcal{M})$ given by

$$\phi''(t) = \left( \phi' \right)_{T_{t} \mathcal{M}} \left. \left( \frac{\partial}{\partial t} \right) \middle|_{t=0} \right).$$

QED

**Proposition 3.6.** Let $F : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ be a smooth map such that $F(0, x) = x$ for all $x \in \mathcal{M}$. For each $x \in \mathcal{M}$ let $F_{x} : \mathbb{R} \to \mathcal{M}$ be the smooth curve defined by $t \mapsto F(t, x)$, so that $F_{x}(0) = x$. Suppose that $F_{x}'(0) = 0_{x}$ for all $x \in \mathcal{M}$, where $0_{x}$ is the origin of the vector space $T_{x} \mathcal{M}$. Then

$$d^{2} \frac{d}{dt^{2}} \bigg|_{t=0} F_{x}$$

defines a vector field on $\mathcal{M}$.

**Proof.** Let $x \in \mathcal{M}$. The velocity vector field of $F_{x}$ is a map $F_{x}' : \mathbb{R} \to T \mathcal{M}$, with $F_{x}'(t) \in T_{F_{x}(t,x)} \mathcal{M}$. By taking another derivative, we obtain a map $F_{x}'' : \mathbb{R} \to T(\mathcal{T} \mathcal{M})$ given by

$$F_{x}''(t_{0}) = (F_{x}')_{T_{x} \mathcal{M}} \left( \frac{\partial}{\partial t} \right)_{T_{x} \mathcal{M}} \bigg|_{t=0} \in T_{F_{x}'(t_{0})}(\mathcal{T} \mathcal{M}).$$

By taking the derivative at $t = 0$ and allowing $x \in \mathcal{M}$ to vary, we have defined a smooth map

$$d^{2} \frac{d}{dt^{2}} \big|_{t=0} F : \mathcal{M} \to T(\mathcal{T} \mathcal{M})$$

with

$$d^{2} \frac{d}{dt^{2}} \bigg|_{t=0} F(t, x) \in T_{F_{x}'(0)}(\mathcal{T} \mathcal{M}).$$
Let $z: \mathcal{M} \to T\mathcal{M}$ be the zero section of the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$. By assumption we have $F'_x(0) = 0_x$ for all $x \in \mathcal{M}$, so $F''_x(0) \in T_{F'_x(0)}(T\mathcal{M}) = T_{0_x}(T\mathcal{M})$ is in the range of $z_{*x}$. Hence we can define a map $\mathcal{M} \to T(T\mathcal{M}) \to T\mathcal{M}$ by

$$x \mapsto (z_{*x})^{-1} \circ \left( \frac{d^2}{dt^2} \bigg|_{t=0} F(t, x) \right).$$

Note that $x$ is mapped to an element in $(z_{*x})^{-1}(T_{F'_x(0)}(T\mathcal{M})) = (z_{*x})^{-1}(T_{0_x}(T\mathcal{M})) = T_x\mathcal{M}$, so this is actually a vector field. For simplicity, although it can be ambiguous, we sometimes drop the zero section from the expression, and denote this vector field on $\mathcal{M}$ by

$$\frac{d^2}{dt^2} \bigg|_{t=0} F.$$

QED

**Lemma 3.7.** Let $X$ and $Y$ be in $\mathfrak{X}(\mathcal{M})$, and let $\Phi^t_X$ and $\Phi^t_Y$ be their respective flows at time $t$. Then

$$\frac{d^2}{dt^2} \bigg|_{t=0} \Phi^{-t}_Y \circ \Phi^{-t}_X \circ \Phi^t_Y \circ \Phi^t_X = 0.$$

**Proof.** See Spivak’s *Comprehensive Introduction to Differential Geometry, Volume One, Third Edition*, Proposition 15 in Chapter 5, on page 160. QED

To paraphrase Spivak, out of the context of Lie groups, it is not clear how anyone ever could have come up with the next theorem. It relates the commutator in the Lie algebra of vector fields on $\mathcal{M}$, the Lie bracket, to the commutator in the group of diffeomorphisms on $\mathcal{M}$, $(f, g) \mapsto g^{-1} \circ f^{-1} \circ g \circ f$.

**Theorem 3.8.** Let $X$ and $Y$ be in $\mathfrak{X}(\mathcal{M})$, and let $\Phi^t_X$ and $\Phi^t_Y$ be their respective flows at time $t$. Then

$$[X, Y] = \frac{1}{2} \frac{d^2}{dt^2} \bigg|_{t=0} \Phi^{-t}_Y \circ \Phi^{-t}_X \circ \Phi^t_Y \circ \Phi^t_X.$$

**Proof.** See Spivak’s *Comprehensive Introduction to Differential Geometry, Volume One, Third Edition*, Theorem 16 in Chapter 5, on page 162. QED

The right side of the equation in Theorem 3.8 defines a vector field on $\mathcal{M}$ due to Proposition 3.6 and Lemma 3.7.
One informal way of thinking of the derivative of a map is as the best local linear approximation. This should mean that if we take the derivative of a linear map, we should, in some sense, obtain the same linear map with which we started. The following lemma makes this idea more precise. We will be using it extensively later.

Lemma 3.9. Let \( V \) and \( W \) be vector spaces, and let \( v \in V \). Suppose that \( L: V \to W \) is a linear map, and let \( \Gamma_v: V \to T_vV \) and \( \Gamma_{L(v)}: W \to T_{L(v)}W \) be the canonical linear isomorphisms. Then the following diagram commutes.

\[
\begin{array}{ccc}
V & \rightarrow & T_vV \\
\downarrow_L & & \downarrow_{T_vL} \\
W & \rightarrow & T_{L(v)}W
\end{array}
\]

4 Symplectic and Hamiltonian Vector Fields

Let \((\mathcal{M}, \omega)\) be a symplectic manifold.

Definition 4.1. Let \( h: \mathcal{M} \to \mathbb{R} \) be a smooth function. Then the symplectic gradient of \( h \) is the unique vector field \( \nabla_\omega h \in \mathfrak{X}(\mathcal{M}) \) such that

\[
dh = \iota(\nabla_\omega h)\omega.
\]

 Existence and uniqueness of such a vector field follow from the non-degeneracy of \( \omega \).

A vector field \( X \in \mathfrak{X}(\mathcal{M}) \) is symplectic if \( \iota(X)\omega \) is closed, and \( X \) is Hamiltonian if it is the symplectic gradient of some smooth function in \( C^\infty(\mathcal{M}) \).

Lemma 4.2. If \( X \in \mathfrak{X}(\mathcal{M}) \) is Hamiltonian, then it is symplectic.

Proof. If \( X \) is Hamiltonian, then there is some \( h \in C^\infty(\mathcal{M}) \) such that \( X = \nabla_\omega h \). Then by the definition of \( \nabla_\omega \) we have

\[
d\iota(X)\omega = d\iota(\nabla_\omega h)\omega = d(dh) = 0,
\]

since \( d^2 = 0 \). Hence \( X \) is symplectic. QED

Proposition 4.3. Let \( X \in \mathfrak{g}(\mathcal{M}) \), and let \( \phi_t \) be the flow of \( X \). The following are equivalent.
1. $\iota(X)\omega$ is closed ($X$ is symplectic);

2. $\mathcal{L}_X\omega = 0$;

3. $\phi^*_t\omega = \omega$ for all $t$ (the flow of $X$ preserves the symplectic structure of $\mathcal{M}$).

**Proof.** Statements 2 and 3 above are obviously equivalent, since one is obtained immediately from the other by differentiation or integration. We will prove that 1 is equivalent to 2.

By Cartan’s identity, we know that

$$\mathcal{L}_X = d \circ \iota(X) + \iota(X) \circ d.$$

Because the symplectic form $\omega$ is by definition closed, we have

$$\mathcal{L}_X\omega = d(\iota(X)\omega) + \iota(X)d\omega = d(\iota(X)\omega) + \iota(X)(0) = d(\iota(X)\omega).$$

Thus statements 1 and 2 are equivalent. QED

**Lemma 4.4.** Let $X, Y \in \mathfrak{X}(\mathcal{M})$ be symplectic vector fields. Then $[X, Y]$ is a Hamiltonian vector field.

**Proof.** Let $f: \mathcal{M} \to \mathbb{R}$ be defined by

$$f = \iota(X)\iota(Y)\omega.$$

To prove that $\nabla_\omega f = [X, Y]$, by definition we must show that $df = \iota([X, Y])\omega$. Using parts (iv) and (ii) of Theorem 3.1 and the fact that $\omega$ is closed, we have

$$df = d(\iota(X)\iota(Y)\omega) = \mathcal{L}_X\iota(Y)\omega - \iota(X)(\iota(Y)d\omega)$$

$$= \mathcal{L}_X\iota(Y)\omega - \iota(X)(\text{lied}_Y \omega - \iota(Y)d\omega) = \mathcal{L}_X\iota(Y)\omega - \iota(X)(\text{lied}_Y \omega)$$

$$= (\mathcal{L}_X\iota(Y) - \iota(X)\mathcal{L}_Y)\omega = \iota([X, Y])\omega.$$

QED

**Corollary 4.5.** The set $\text{sp}(\mathcal{M})$ of symplectic vector fields of $\mathcal{M}$ is a Lie algebra.
Proof. Certainly \( \mathbb{R} \)-linear combinations of symplectic vector fields are symplectic, since the interior product \( \iota \) and the exterior derivative \( d \) are both \( \mathbb{R} \)-linear. By Lemma 4.4, the Lie bracket of two symplectic vector fields is Hamiltonian. By Lemma 4.2, all Hamiltonian vector fields are symplectic. Therefore the Lie bracket of two symplectic vector fields is symplectic, and hence \( \mathfrak{sp}(\mathcal{M}) \) is a Lie algebra. QED

Proposition 4.6. Let \((\mathcal{M}, \omega)\) be a simply-connected symplectic manifold. Then every symplectic vector field is Hamiltonian.

Proof. By definition, we have \( \pi_1(\mathcal{M}) = 0 \). Since \( H_1(\mathcal{M}) \) is the abelianization of \( \pi_1(\mathcal{M}) \), this implies that \( H_1(\mathcal{M}) = 0 \). The Universal Coefficient Theorem for Cohomology gives us the short exact sequence

\[
0 \to \text{Ext}(H_0(\mathcal{M}), \mathbb{R}) \to H^1(\mathcal{M}; \mathbb{R}) \to \text{Hom}(H_1(\mathcal{M}), \mathbb{R}) \to 0.
\]

Since \( H_1(\mathcal{M}) = 0 \), we know \( \text{Hom}(H_1(\mathcal{M}), \mathbb{R}) = 0 \). The group \( \text{Ext}(H_0(\mathcal{M}), \mathbb{R}) \) is 0 if \( H_0(\mathcal{M}) \) is free, which it always is. Therefore we have an even shorter short exact sequence

\[
0 \to H^1(\mathcal{M}; \mathbb{R}) \to 0,
\]

and thus \( H^1(\mathcal{M}; \mathbb{R}) = 0 \). By the De Rham Theorem, we know that

\[
0 = H^1(\mathcal{M}; \mathbb{R}) \cong \frac{\{\text{closed 1-forms on } \mathcal{M}\}}{\{\text{exact 1-forms on } \mathcal{M}\}}.
\]

This means exactly that every closed 1-form on \( \mathcal{M} \) is also exact.

Suppose \( X \) is a symplectic vector field, so the 1-form \( \iota(X)\omega \) is closed. Then this 1-form is also exact, so there exists a 0-form \( h \in C^\infty(\mathcal{M}) \) such that \( dh = \iota(X)\omega \). This means precisely that \( \nabla_\omega h = X \), so \( X \) is Hamiltonian. QED

5 Poisson Brackets

Let \((\mathcal{M}, \omega)\) be a symplectic manifold.

Definition 5.1. Let \( f, g \in C^\infty(\mathcal{M}) \). Then the **Poisson bracket** of \( f \) and \( g \) is the element \( \{f, g\} \in C^\infty(\mathcal{M}) \) defined by

\[
\{f, g\} := \iota(\nabla_\omega f)\iota(\nabla_\omega g)\omega.
\]
This means that
\[ \{f, g\}(x) = \omega((\nabla_\omega g)(x), (\nabla_\omega f)(x)) \]
for all \( x \in \mathcal{M} \). \( \triangle \)

**Proposition 5.2.** (1) The Poisson bracket makes \( C^\infty(\mathcal{M}) \) into a Lie algebra.

(2) For all \( f, g \in C^\infty(\mathcal{M}) \),
\[ [\nabla_\omega f, \nabla_\omega g] = \nabla_\omega \{f, g\} \]

(3) The map \( C^\infty(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}) \) given by \( f \mapsto \nabla_\omega f \) is a Lie algebra homomorphism. (Of course, the symplectic gradient of a function is always a symplectic vector field, so we actually have a Lie algebra homomorphism \( C^\infty(\mathcal{M}) \to \mathfrak{sp}(\mathcal{M}). \))

**Proof.** Since the interior product \( \iota \) is linear, to prove (1) we need only show that the Poisson bracket satisfies the Jacobi identity. This is straightforward, but messy.

The proof of (2) is essentially the same as the proof of Lemma 4.4.

Statement (3) follows from statement (2) and the fact that the symplectic gradient \( \nabla_\omega \) is linear. QED

**Claim 5.3.** Let \( f, g \in C^\infty(\mathcal{M}) \). Then
\[ \{f, g\} := \iota(\nabla_\omega f)\iota(\nabla_\omega g)\omega = df(\nabla_\omega g) = -dg(\nabla_\omega f). \]

**Proof.** This follows quickly from the definition of the symplectic gradient. QED

## 6 Smooth and Symplectic Actions

Let \( \mathcal{M} \) be a manifold and \( G \) be a Lie group. A **smooth action** of \( G \) on \( \mathcal{M} \) is a smooth map
\[ \lambda: G \times \mathcal{M} \to \mathcal{M} \]
such that

1. \( \lambda(e, x) = x \) for all \( x \in \mathcal{M} \), where \( e \in G \) is the identity element; and
2. \( \lambda(g_1 g_2, x) = \lambda(g_1, \lambda(g_2, x)) \) for all \( g_1, g_2 \in G \) and \( x \in \mathcal{M} \).
For each \( g \in G \), we define a map \( \lambda^g : M \to M \) by \( x \mapsto g \cdot x := \lambda(g, x) \). Note that this map is smooth. It is actually a diffeomorphism. If \( g^{-1} \) is the inverse of \( g \), then for each \( x \in M \) we have

\[
\lambda^{g^{-1}} \circ \lambda^g(x) = \lambda(g^{-1}, \lambda(g, x)) = \lambda(g^{-1}g, x) = \lambda(e, x) = x
\]

and

\[
\lambda^g \circ \lambda^{g^{-1}}(x) = \lambda(g, \lambda(g, x)) = \lambda(gg^{-1}, x) = \lambda(e, x) = x,
\]

so \( \lambda g^{-1} \) is a smooth inverse of \( \lambda g \). We thus have a group homomorphism \( G \to \text{Diff}(M) \) given by \( g \mapsto \lambda(g) \).

For each \( x \in M \), we define a map \( \lambda^x : G \to M \) by \( g \mapsto g \cdot x := \lambda(g, x) \). This is a smooth map, and of course \( \lambda_x(g) = \lambda^g(x) = \lambda(g, x) \) for all \( g \in G \) and \( x \in M \).

**Definition 6.1.** For each \( \xi \in g \), the vector field \( \nu_\lambda(\xi) \) on \( M \) defined by

\[
\nu_\lambda(\xi)(x) := \frac{d}{dt} \bigg|_{t=0} \lambda(\exp(t\xi), x) = T_e \lambda^x(\xi)
\]

is called the fundamental vector field on \( M \) induced by \( \xi \). If there is no confusion, we will often write \( \xi_M \) instead of \( \nu_\lambda(\xi) \).

**Proposition 6.2.** Let \( \xi \in g \) and \( x \in M \). Then the flow of \( \xi_M \) through \( x \) is given by

\[
c(t) = \lambda(\exp(t\xi), x).
\]

**Proof.** Let \( t \in \mathbb{R} \). We calculate

\[
\frac{d}{dt} c(t) = \frac{d}{ds} \bigg|_{s=t} c(t) = \frac{d}{ds} \bigg|_{s=t} c(s)
\]

\[
= \frac{d}{ds} \bigg|_{s=t} \lambda(\exp(s\xi), x) = \frac{d}{ds} \bigg|_{s=0} \lambda(\exp((s + t)\xi), x)
\]

\[
= \frac{d}{ds} \bigg|_{s=0} \lambda^{\exp(s\xi)} \circ \lambda^{\exp(t\xi)}(x) = \xi_M(\lambda(\exp t\xi, x)) = \xi_M(c(t)).
\]

QED

**Definition 6.3.** Suppose \( M \) has a symplectic form \( \omega \). If each \( \lambda(g) \) is a symplectomorphism, then the action of \( G \) on \((M, \omega)\) is said to be a symplectic action.
Lemma 6.4. Let \((\mathcal{M}, \omega)\) be a symplectic manifold. The vector field \(\xi_M\) is symplectic for each \(\xi \in \mathfrak{g}\).

Proof. Recall that the flow of \(\xi_M\) is given by

\[
t \mapsto \lambda(\exp(t\xi)).
\]

Since \(\lambda\) is a representation of \(G\) through symplectomorphisms of \(G\), this means that the flow of \(\xi_M\) for each \(t\) is a symplectomorphism. By Proposition 4.3, this means that \(\xi_M\) is a symplectic vector field. QED

Proposition 6.5. The map \(\mathfrak{g} \to \mathfrak{sp}(\mathcal{M})\) given by \(\xi \mapsto \xi_M\) is a Lie algebra homomorphism.

Proof. First we show that this map is linear. Let \(x \in \mathcal{M}\) and \(\xi \in \mathfrak{g}\). Then

\[
\xi_M(x) = (d\lambda_x)_e(\xi).
\]

The right side of this equation is linear in \(\xi\), since derivatives are linear. Hence the left side of the equation is linear in \(\xi\) as well, so the map \(\xi \mapsto \xi_M\) is linear.

Now we prove the hard part. Let \(\xi, \eta \in \mathfrak{g}\). Note that \(\lambda^{\exp(t\xi)}\) and \(\lambda^{\exp(t\eta)}\) are the flows of \(\xi_M\) and \(\eta_M\), respectively. By Theorem 3.8, and the homomorphism property of group representations, we have

\[
[\xi_M, \eta_M] = \frac{1}{2} \frac{d^2}{dt^2} \bigg|_{t=0} \lambda^{\exp(-t\eta)} \circ \lambda^{\exp(-t\xi)} \circ \lambda^{\exp(t\eta)} \circ \lambda^{\exp(t\xi)}
\]

\[
= \frac{1}{2} \frac{d^2}{dt^2} \bigg|_{t=0} \lambda^{\exp(-t\eta) \exp(-t\xi) \exp(t\eta) \exp(t\xi)}
\]

\[
= \frac{1}{2} \frac{d^2}{dt^2} \bigg|_{t=0} \lambda^{\exp(t\eta)^{-1} \exp(t\xi)^{-1} \exp(t\eta) \exp(t\xi)}.
\]

Let \(x \in \mathcal{M}\). Let \(c(t)\) denote the smooth curve

\[
t \mapsto \exp(t\eta)^{-1} \exp(t\xi)^{-1} \exp(t\eta) \exp(t\xi) \in G.
\]

Then

\[
[\xi_M, \eta_M](x) = \frac{1}{2} (\lambda_x \circ c)'(0)
\].

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Let \(z_M : \mathcal{M} \rightarrow T\mathcal{M}\) and \(z_G : G \rightarrow TG\) denote the appropriate zero sections. Then by definition we have
\[
(\lambda_x \circ c)'(0) = (\lambda_x \circ c)_* \circ \frac{\partial}{\partial t}(0) = (\lambda_x)_* \circ c_* \circ \frac{\partial}{\partial t}(0),
\]
so
\[
(\lambda_x \circ c)''(0) = \left( (\lambda_x)_* \circ c_* \circ \frac{\partial}{\partial t} \right)_* \circ \frac{\partial}{\partial t}(0).
\]
Using the commutative diagram in Remark 3.3, this last line is equal to
\[
(\lambda_x)_* \circ (z_G)^{-1} \circ c_* \circ \left( \frac{\partial}{\partial t} \right)_* \circ \frac{\partial}{\partial t}(0) = (\lambda_x)_* \circ c''(0).
\]
Thus
\[
[\xi_M, \eta_M](x) = \frac{1}{2} (\lambda_x \circ c)''(0)
\]
\[
= \frac{1}{2} (\lambda_x)_* \circ c''(0) = (\lambda_x)_* \circ \frac{1}{2} c''(0).
\]
Using the identification of the Lie algebra \(\mathfrak{g}\) of \(G\) with left-invariant vector fields, we can use Theorem 3.8 again to see that
\[
\frac{1}{2} c''(0) = [\xi, \eta] \in \mathfrak{g},
\]
so
\[
[\xi_M, \eta_M](x) = (\lambda_x)_* \circ \frac{1}{2} c''(0) = (\lambda_x)_*[\xi, \eta] = (d\lambda_x)_e[\xi, \eta] = ([\xi, \eta])_\mathcal{M}(x).
\]
QED

**Lemma 6.6.** Let \(G\) be a Lie group acting symplectically on a symplectic manifold \((\mathcal{M}, \omega)\). If \([\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}\), then the induced vector field \(\xi_M\) is Hamiltonian, for each \(\xi \in \mathfrak{g}\).

**Proof.** Let \(\xi \in \mathfrak{g}\). If \([\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}\), then there are \(X, Y \in \mathfrak{g}\) such that \([X, Y] = \xi\). By Lemma 6.4, we know that \(X_M\) and \(Y_M\) are symplectic vector fields. By Lemma 4.4, we know that \([= Y_M, X_M]\) is a Hamiltonian vector field. Finally, Proposition 6.5 implies that \(\xi_M = [X, Y]_\mathcal{M} = [Y_M, X_M]\) is a Hamiltonian vector field. QED

Let \(G\) be a Lie group. Recall from Section that \(G\) acts on \(\mathfrak{g}\) by the adjoint action \(\text{Ad} : G \rightarrow \mathfrak{g}\) and on \(\mathfrak{g}^*\) by the coadjoint action \(\text{Ad}^* : G \rightarrow \mathfrak{g}^*\). Thus for each \(X \in \mathfrak{g}\) and \(\xi \in \mathfrak{g}^*\) we have the fundamental vector fields \(\nu_{\text{Ad}}(X) \in \mathfrak{X}(\mathfrak{g})\) and \(\nu_{\text{Ad}^*}(\xi) \in \mathfrak{X}(\mathfrak{g}^*)\).

**Proposition 6.7.** Let \(X, Y \in \mathfrak{g}\) and \(\xi \in \mathfrak{g}^*\). Let \(\Gamma_Y : \mathfrak{g} \rightarrow T_Y \mathfrak{g}\) and \(\Gamma_\xi : \mathfrak{g}^* \rightarrow T_\xi \mathfrak{g}^*\) be the canonical linear isomorphisms.
(1) $\nu_{Ad}(X)(Y) = \Gamma_Y([X,Y])$.

(2) $[\Gamma^{-1}_\xi \circ \nu_{Ad^r}(X)(\xi)](Y) = \xi([Y,X])$.

**Proof.** Part (1) follows immediately from the definition

$$\nu_{Ad}(X)(Y) := \frac{d}{dt} \bigg|_{t=0} Ad(\exp tX)(Y)$$

and from Proposition 2.6.

For Part (2), let $F: g^* \to \mathbb{R}$ be given by $F(\eta) = \eta(Y)$ for each $\eta \in g^*$. Note that $F$ is a linear map. Then using Lemma 3.9 several times, we calculate

$$[\Gamma^{-1}_\xi \circ \nu_{Ad^r}(X)(\xi)](Y) = F \circ \Gamma^{-1}_\xi(\nu_{Ad^r}(X)(\xi))$$

$$= F \circ \Gamma^{-1}_\xi \left( \frac{d}{dt} \bigg|_{t=0} Ad^r(\exp tX)(\xi) \right) = \Gamma^{-1}_\xi \left( \frac{d}{dt} \bigg|_{t=0} Ad^r(\exp tX)(\xi) \right)$$

$$= \Gamma^{-1}_\xi \left( \frac{d}{dt} \bigg|_{t=0} F(Ad^r(\exp tX)(\xi)) \right) = \Gamma^{-1}_\xi \left( \frac{d}{dt} \bigg|_{t=0} (Ad^r(\exp tX)(\xi))(Y) \right)$$

$$= \Gamma^{-1}_\xi \left( \frac{d}{dt} \bigg|_{t=0} \xi(Ad(\exp(-tX))(Y)) \right) = \Gamma^{-1}_\xi \left( \frac{d}{dt} \bigg|_{t=0} \xi(Ad(\exp(-tX))(Y)) \right)$$

$$= \Gamma^{-1}_\xi \circ \xi^* \circ \Gamma_Y([Y,X]) = \Gamma^{-1}_\xi \circ \xi^* \circ \Gamma_Y([Y,X])$$

$$= \xi([Y,X]).$$

QED

7 **Equivariance**

Let $\lambda: G \to \mathcal{M}$ be a symplectic action of a Lie group $G$ on a symplectic manifold $\mathcal{M}$ with symplectic form $\omega$. Then $G$ also acts on itself, and $g$, and $\mathfrak{g}^*$, and $C^\infty(\mathcal{M})$.

As detailed in Section , $G$ acts on itself by conjugation, and $G$ acts on $\mathfrak{g}$ by the adjoint action.

The action of $G$ on $C^\infty(\mathcal{M})$ is essentially the dual of the action $\lambda$, denoted $\lambda^* : G \to C^\infty(\mathcal{M})$, and given by

$$\lambda^*(g)(f) := f \circ \lambda^* g^{-1},$$

for $g \in G$ and $f \in C^\infty(\mathcal{M})$.  

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**Definition 7.1.** Let $\mu: \mathcal{M} \to \mathfrak{g}^*$ and $\mathcal{H}: \mathfrak{g} \to C^\infty(\mathcal{M})$ be smooth maps, where $\mathcal{H}$ is smooth in the sense that the associated map $\mathfrak{g} \times \mathcal{M} \to \mathbb{R}$, $(\xi, x) \mapsto \mathcal{H}(\xi)(x)$ is smooth. Then $\mu$ is **equivariant** with respect to the actions of $G$ on $\mathcal{M}$ and $\mathfrak{g}^*$ if the following diagram commutes for all $g \in G$.

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\mu} & \mathfrak{g}^* \\
\downarrow{\lambda^g} & & \downarrow{\text{Ad}^*(g)} \\
\mathcal{M} & \xrightarrow{\mu} & \mathfrak{g}^*
\end{array}
\]

The map $\mathcal{H}$ is **equivariant** with respect to the actions of $G$ on $\mathfrak{g}$ and $C^\infty(\mathcal{M})$ if the following diagram commutes for all $g \in G$.

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\mathcal{H}} & C^\infty(\mathcal{M}) \\
\downarrow{\text{Ad}(g)} & & \downarrow{\lambda^*(g)} \\
\mathfrak{g} & \xrightarrow{\mathcal{H}} & C^\infty(\mathcal{M})
\end{array}
\]

Since $\mu$ is smooth, so is the associated map $\mathcal{M} \times \mathfrak{g} \to \mathbb{R}$, $(x, \xi) \mapsto \mu(x)(\xi)$. Changing the order of the Cartesian product in this domain gives a smooth map $\mathfrak{g} \times \mathcal{M} \to \mathbb{R}$, $(\xi, x) \mapsto \mu(x)(\xi)$. Associated to this, then, is a map $\mathfrak{g} \to C^\infty(\mathcal{M})$ that is smooth in the same sense that $\mathcal{H}: \mathfrak{g} \to C^\infty(\mathcal{M})$ above is smooth. This last map is the **transpose** of $\mu$, denoted $\mu^T: \mathfrak{g} \to C^\infty(\mathcal{M})$.

In an entirely analogous way, the map $\mathcal{H}: \mathfrak{g} \to C^\infty(\mathcal{M})$ induces a smooth map $\mathcal{M} \to \mathfrak{g}^*$, called the **transpose** of $\mathcal{H}$, denoted $\mathcal{H}^T: \mathcal{M} \to \mathfrak{g}^*$.

**Remark 7.2.** If $\alpha \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$, it is common to denote the real number $\alpha(\xi)$ by $\langle \alpha, \xi \rangle$. This is known as the **canonical pairing** between elements of $\mathfrak{g}^*$ and $\mathfrak{g}$. With this convention, for $\xi \in \mathfrak{g}$, the element $\mu^T(\xi) \in C^\infty(\mathcal{M})$ is sometimes denoted $\langle \mu, \xi \rangle$.

To recap, if $\xi \in \mathfrak{g}$, then $\langle \mu, \xi \rangle \in C^\infty(\mathcal{M})$ is given by

\[\langle \mu, \xi \rangle(x) = \langle \mu(x), \xi \rangle = \mu(x)(\xi)\]

for all $x \in \mathcal{M}$.

**Proposition 7.3.** Let $\mu: \mathcal{M} \to \mathfrak{g}^*$ and $\mathcal{H}: \mathfrak{g} \to C^\infty(\mathcal{M})$ be smooth maps. Then

1. $\mu$ is equivariant if and only if $\mu^T$ is equivariant.
(2) $\mathcal{H}$ is equivariant if and only if $\mathcal{H}^T$ is equivariant.

**Proof.** Note that (1) and (2) are essentially equivalent. We will only prove (1). Let $\Phi: \mathcal{M} \times \mathfrak{g} \to \mathbb{R}$ be given by
$$\Phi(x, \xi) = \mu(x)(\xi)$$
for all $(x, \xi) \in \mathcal{M} \times \mathfrak{g}$.

Suppose first that $\mu$ is equivariant. This means precisely that
$$\mu(\lambda^g(x))(\xi) = \text{Ad}^*(g)(\mu(x))(\xi)$$
for all $g \in G, x \in \mathcal{M}, \xi \in \mathfrak{g}$. Note that $\mu(\lambda^g(x))(\xi) = \Phi(\lambda^g(x), \xi)$ and $\text{Ad}^*(g)(\mu(x))(\xi) = \Phi(x, \text{Ad}(g^{-1})(\xi))$. Thus $\mu$ is equivariant if and only if
$$\Phi(\lambda^g(x), \xi) = \Phi(x, \text{Ad}(g^{-1})(\xi))$$  \hspace{1cm} (1)
for all $g \in G, x \in \mathcal{M}, \xi \in \mathfrak{g}$.

Suppose that $\mu^T$ is equivariant. This means precisely that
$$\mu^T(\text{Ad}(g)(\xi))(x) = \lambda^*(g)(\mu^T(\xi))(x)$$
for all $g \in G, x \in \mathcal{M}, \xi \in \mathfrak{g}$. Note that $\mu^T(\text{Ad}(g)(\xi))(x) = \Phi(x, \text{Ad}(g)(\xi))$ and $\lambda^*(g)(\mu^T(\xi))(x) = \Phi(\lambda^{-1}(x), \xi)$. Thus $\mu$ is equivariant if and only if
$$\Phi(x, \text{Ad}(g)(\xi)) = \Phi(\lambda^g(x), \xi)$$  \hspace{1cm} (2)
for all $g \in G, x \in \mathcal{M}, \xi \in \mathfrak{g}$.

Because every element $g \in G$ has an inverse $g^{-1}$, Equation 1 holds for all $g \in G, x \in \mathcal{M}, \xi \in \mathfrak{g}$ if and only if Equation 2 holds for all $g \in G, x \in \mathcal{M}, \xi \in \mathfrak{g}$. Therefore $\mu$ is equivariant if and only if $\mu^T$ is equivariant. QED

**Definition 7.4.** If the action $\lambda$ of $G$ on $\mathcal{M}$ is Hamiltonian, then each $\xi \in \mathfrak{g}$ corresponds to a vector field $\xi_\mathcal{M} \in \mathfrak{sp}(\mathcal{M})$, which corresponds to some (not necessarily unique) $h \in C^\infty(\mathcal{M})$ such that
$$\nabla_\omega h = \xi_\mathcal{M}.$$  

This defines a (not necessarily unique) map $\mathcal{H}: \mathfrak{g} \to C^\infty(\mathcal{M})$ such that
$$\nabla_\omega \mathcal{H}(\xi) = \xi_\mathcal{M}$$

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for all $\xi \in \mathfrak{g}$. Note that $\mathcal{H}$ is linear. Such a map $\mathcal{H}$ which is also smooth will be called a **comoment map** of the symplectic action $\lambda$ of $G$ on $\mathcal{M}$. Clearly if a symplectic action has a comoment, then the action is Hamiltonian. △

**Claim 7.5.** Suppose $\lambda$ is a Hamiltonian action. Then there exists a comoment map $\mathcal{H}: \mathfrak{g} \to C^\infty(\mathcal{M})$.

**Proof.** Let $\{\xi_1, \ldots, \xi_k\}$ be a basis for $\mathfrak{g}$, and let $\{\xi^*_1, \ldots, \xi^*_k\}$ be the corresponding dual basis for $\mathfrak{g}^*$. Since the action is Hamiltonian, there are $f_1, \ldots, f_k \in C^\infty(\mathcal{M})$ such that

$$\nabla_\omega f_i = (\xi_i)_\mathcal{M}$$

for each $i = 1, \ldots k$. Let $\mathcal{H}: \mathfrak{g} \to C^\infty(\mathcal{M})$ be given by

$$\mathcal{H} = \sum_{i=1}^k f_i \xi^*_i,$$

meaning that $\mathcal{H}(\eta)(x) = \sum_{i=1}^k f_i(x) \xi^*_i(\eta)$ for all $\eta \in \mathfrak{g}$ and $x \in \mathcal{M}$. Note that $\mathcal{H}$ is both linear and smooth, by construction.

Let $\eta = \sum_{i=1}^k a_i \xi_i$ be any element in $\mathfrak{g}$, where $a_1, \ldots, a_k \in \mathbb{R}$. Since the operations of taking the symplectic gradient and obtaining the fundamental vector field of an action are both linear, we calculate

$$\nabla_\omega \mathcal{H}(\eta) = \nabla_\omega \left( \sum_{i=1}^k f_i \xi^*_i(\eta) \right) = \nabla_\omega \left( \sum_{i=1}^k f_i a_i \right)$$

$$= \sum_{i=1}^k a_i \nabla_\omega f_i = \sum_{i=1}^k a_i (\xi_i)_\mathcal{M} = \left( \sum_{i=1}^k a_i \xi_i \right)_\mathcal{M} = \eta_\mathcal{M}.$$

QED

**Theorem 7.6.** Let $\mathcal{H}: \mathfrak{g} \to C^\infty(\mathcal{M})$ be a comoment map. Suppose $\mathcal{H}$ is equivariant. Then $\mathcal{H}$ is a Lie algebra homomorphism from $\mathfrak{g}$ with its standard bracket to $C^\infty(\mathcal{M})$ with the Poisson bracket.

**Proof.** Let $x \in \mathcal{M}$ and $\xi, \eta \in \mathfrak{g}$.
Let \( c_2 : \mathbb{R} \to \mathfrak{g} \) be given by \( t \mapsto \text{Ad}(\exp(t\xi))(\eta) \). Note that \( \mathcal{H}^T(x) \) is a map \( \mathfrak{g} \to \mathbb{R} \). Let \( c_1 : \mathbb{R} \to \mathbb{R} \) be given by \( t \mapsto ((\mathcal{H}^T(x)) \circ c_2)(t) \). Let

\[
\Gamma_Y : \mathfrak{g} \to T_Y \mathfrak{g} \quad \text{and} \quad \Gamma_{\mathcal{H}^T(x)(\eta)} : \mathbb{R} \to T_{\mathcal{H}^T(x)(\eta)} \mathbb{R}
\]

be the canonical linear isomorphisms.

By Proposition 2.6, we have \( c_2(0) = \Gamma_Y([\xi, \eta]) \). Then using Lemmas 3.5 and 3.9, we calculate

\[
c_1(0) = ((\mathcal{H}^T(x)) \circ c_2)'(0) = (\mathcal{H}^T(x))_{*,\eta}(c_2(0)) = (\mathcal{H}^T(x))_{*,\eta}(\Gamma_Y([\xi, \eta])) = \Gamma_{\mathcal{H}^T(x)(\eta)}(\mathcal{H}^T(x)([\xi, \eta]))
\]

Let \( c_4 : \mathbb{R} \to \mathcal{M} \) be given by \( t \mapsto \lambda_x(\exp(-t\xi)) \). Note that \( \mathcal{H}(\eta) \) is a map \( \mathcal{M} \to \mathbb{R} \). Let \( c_3 : \mathbb{R} \to \mathbb{R} \) be given by \( t \mapsto \mathcal{H}(\eta)(c_4(t)) \). Note that \( c_4(0) = (-\xi)_{\mathcal{M}}(x) \). Then using Lemma 3.5 and Claim 5.3 and the defining property of comoment maps, we calculate

\[
c_3(0) = (\mathcal{H}(\eta) \circ c_4)'(0) = [\mathcal{H}(\eta)]_{*,x}((0))
\]

\[
[\mathcal{H}(\eta)]_{*,x}((-\xi)_{\mathcal{M}}(x)) = [\mathcal{H}(\eta)]_{*,x}((\nabla_\omega \mathcal{H}(-\xi))(x)) = \Gamma_{\mathcal{H}(\eta)(x)}\left(dH(\nabla_\omega \mathcal{H}(-\xi))(x)\right)
\]

Because \( \mathcal{H} \) is equivariant, for all \( g \in G, v \in \mathfrak{g}, p \in \mathcal{M}, \) we have

\[
\mathcal{H}(\text{Ad}(g)v)(p) = \mathcal{H}(v)(\lambda_{g^{-1}}(p)) = \mathcal{H}(v)(\lambda_p(g^{-1}))
\]

Let \( t \in \mathbb{R} \). Then

\[
c_1(t) = ((\mathcal{H}^T(x)) \circ c_2)(t) = (\mathcal{H}^T(x))(\text{Ad}(\exp(t\xi))(\eta)) = \mathcal{H}(\eta) \circ \lambda_x(\exp(-t\xi)) = \mathcal{H}(\eta)(c_4(t)) = c_3(t).
\]

Therefore

\[
\Gamma_{\mathcal{H}(\eta)(x)}(\mathcal{H}([\xi, \eta])(x)) = c_1'(0) = c_3'(0) = \Gamma_{\mathcal{H}(\eta)(x)}\left(\{\mathcal{H}(\xi), \mathcal{H}(\eta)\}(x)\right).
\]
Since $\Gamma_{\mathcal{H}(\eta)}(x)$ is an isomorphism, this implies that

$$\mathcal{H}([\xi, \eta])(x) = \{\mathcal{H}(\xi), \mathcal{H}(\eta)\}(x).$$

QED

The converse of Theorem is true if the Lie group $G$ is connected. Before we prove this, we need a lemma.

**Lemma 7.7.** Let $\mathcal{H}: g \to C^\infty(M)$ be a comoment map, and put $\mu = \mathcal{H}^T: M \to g^*$. Let $x \in M$ and $Z \in T_x M$ and $Y \in g$, and let $\Gamma_{\mu(x)}: g^* \to T_{\mu(x)}g^*$ be the canonical linear isomorphism. Then

$$\Gamma_{\mu(x)}^{-1}(\mu_*(x)(Z))(Y) = \omega_x(Y_M(x), Z).$$

**Proof.** Let $c: \mathbb{R} \to M$ be a smooth curve with $c(0) = x$ and $\dot{c}(0) = Z$, and let $F: g^* \to \mathbb{R}$ be given by $F(\eta) = \eta(Y)$ for each $\eta \in g^*$. Let $\Gamma_{\mu(x)(Y)}: \mathbb{R} \to T_{\mu(x)(Y)}\mathbb{R}$ be the canonical linear isomorphism. Then using Lemma 3.9 several times, we calculate

$$\Gamma_{\mu(x)}^{-1}(\mu_*(x)(Z))(Y) = F \circ \Gamma_{\mu(x)}^{-1}(\mu_*(x)(\frac{d}{dt}|_{t=0}c(t))) = F \circ \Gamma_{\mu(x)}^{-1}((\frac{d}{dt}|_{t=0}\mu(c(t))))$$

$$= F \circ \Gamma_{\mu(x)}^{-1}(\mu_*(x)(\frac{d}{dt}|_{t=0}\mu(c(t)))) = \Gamma_{\mu(x)(Y)}^{-1}(F(\mu(c(t)))) = \Gamma_{\mu(x)(Y)}^{-1}(\frac{d}{dt}|_{t=0}\mathcal{H}(Y)(c(t)))$$

$$= \Gamma_{\mu(x)(Y)}^{-1}(\mathcal{H}(Y)_*x(\frac{d}{dt}|_{t=0}\mu(c(t)))) = \Gamma_{\mu(x)(Y)}^{-1}(\mathcal{H}(Y)(c(t)))$$

$$= (d\mathcal{H}(Y))(Z) = \omega_x((\nabla_{\cdot} \mathcal{H}(Y))(x), Z)$$

$$= \omega_x(Y_M(x), Z).$$

QED

**Theorem 7.8.** Suppose $G$ is connected, and $\mu: M \to g^*$ is a smooth map such that $\mathcal{H} = \mu^T: g \to C^\infty(M)$ is a comoment map and a Lie algebra homomorphism. Then $\mu$ is equivariant.
Proof. First we prove the infinitesimal equivariance formula,

\[ \mu_{*,x}(X_M(x)) = \left( \nu_{\text{Ad}^*}(X) \right)(\mu(x)), \]

where \( x \in M \) and \( X \in \mathfrak{g} \).

Let \( x \in M \), and \( X, Y \in \mathfrak{g} \), and let \( \Gamma_{\mu(x)} : \mathfrak{g}^* \to T_{\mu(x)}G^* \) be the canonical linear isomorphism. Using Lemma 7.7 and Proposition 6.7, we calculate

\[
\Gamma_{\mu(x)}^{-1} \left( \mu_{*,x}(X_M(x)) \right)(Y) = \omega_x \left( Y_M(x), X_M(x) \right) \\
= \omega_x \left( \left( \nabla_\omega \mathcal{H}(Y) \right)(x), \left( \nabla_\omega \mathcal{H}(X) \right)(x) \right) = \{ \mathcal{H}(Y), \mathcal{H}(X) \}(x) \\
= \mathcal{H}([Y, X])(x) = \mu(x)([Y, X]) \\
= \left( \nu_{\text{Ad}^*}(X) \right)(\mu(x)).
\]

Since \( Y \in \mathfrak{g} \) was chosen arbitrarily and \( \Gamma_{\mu(x)} \) is a linear isomorphism, this proves the desired identity.

Now, because \( G \) is connected, any neighborhood in \( G \) of the identity generates all of \( G \). The image of the exponential map \( \exp : \mathfrak{g} \to G \) contains is certainly one such neighborhood, so every \( g \in G \) can be written as a finite product of elements of the form \( \exp(X) \) for \( X \in \mathfrak{g} \). Therefore we only need to prove that \( \mu \) is equivariant with respect to the actions of elements of \( G \) of this form, and we will have proved invariance with respect to the actions of all elements of \( G \).

Let \( X \in \mathfrak{g} \). Define \( c : \mathbb{R} \to G \) by \( t \mapsto \exp(tX) \). By Proposition 6.2, we know that for each \( x \in M \), \( \lambda(c(t), x) \) is an integral curve of \( X_M \) through \( x \) and \( \text{Ad}^*(c(t))(\mu(x)) \) is an integral curve of \( \nu_{\text{Ad}^*}(X) \) through \( \mu(x) \). Using the infinitesimal equivariance formula proved above, we know that

\[
\mu_{*,\lambda(c(t),x)}(X_M(x)) = \left( \nu_{\text{Ad}^*}(X) \right)(\mu(\lambda(c(t),x))).
\]

Thus the curve \( \mu(\lambda(c(t), x)) \) is integral to \( \nu_{\text{Ad}^*}(X) \) through \( \mu(x) \). Since \( \text{Ad}^*(c(t))(\mu(x)) \) also satisfies this condition, the uniqueness of integral curves implies that

\[
\mu(\lambda(c(t), x)) = \text{Ad}^*(c(t))(\mu(x))
\]

for all \( t \in \mathbb{R} \). Taking \( t = 1 \), we obtain \( \mu(\lambda(\exp X, x)) = \text{Ad}^*(\exp X)(\mu(x)) \), so \( \mu \) is equivariant with respect to the actions of \( \exp(X) \).

This completes the proof. QED
8 Moment Maps

Let $(\mathcal{M}, \omega)$ be a symplectic manifold, let $G$ be a Lie group, and let $\lambda: G \times \mathcal{M} \to \mathcal{M}$ be a symplectic action.

**Definition 8.1.** A smooth map $\mu: \mathcal{M} \to g^*$ is a **moment map** of the action $\lambda$ if:

1. $\mu$ is equivariant with respect to the actions of $G$ on $\mathcal{M}$ and $g^*$, and
2. $\nabla_\omega \langle \mu, \xi \rangle = \xi_M$ for all $\xi \in g$, or equivalently, $\mu^T: g \to C^\infty(\mathcal{M})$ is a comoment map for the action $\lambda$.

Theorems 7 and 7.8 and the definition of moment maps lead immediately to the following corollary.

**Corollary 8.2.** (1) If $\mu: \mathcal{M} \to g^*$ is a moment map, then $\mu^T: g \to C^\infty(\mathcal{M})$ is a Lie algebra homomorphism.

(2) If $G$ is connected and $\mathcal{H}: g \to C^\infty(\mathcal{M})$ is a comoment and Lie algebra homomorphism, then $\mathcal{H}^T: \mathcal{M} \to g^*$ is a moment map.

**Theorem 8.3.** Let $G$ be a compact Lie group acting symplectically on a symplectic manifold $(\mathcal{M}, \omega)$. If the induced vector field $\xi_M$ is Hamiltonian for all $\xi \in g$, then there is a moment map $\mu: \mathcal{M} \to g^*$.

**Proof.** By Claim 7.5, there is a comoment map $\mathcal{H}: g \to C^\infty(\mathcal{M})$. Let $\tilde{\mu}: \mathcal{M} \to g^*$ be its transpose. Then

$$\nabla_\omega \langle \tilde{\mu}, \xi \rangle = \nabla_\omega \mathcal{H}(\xi) = \xi_M$$

for all $\xi \in g$. Equivalently, by the definition of the symplectic gradient, we have

$$d\langle \tilde{\mu}, \xi \rangle = \iota(\xi_M)\omega$$

for all $\xi \in g$. Equivalently, for all $\xi \in g$, $x \in \mathcal{M}$, and $v \in T_x\mathcal{M}$, we have

$$(d\langle \tilde{\mu}, \xi \rangle)_x v = \omega_x(\xi_M(x), v).$$

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Choose a left invariant Haar measure $\gamma$ on $G$, scaled so that $\gamma(G) = 1$. We can do this because $G$ is compact. Define a new map $\mu : \mathcal{M} \to g^*$ by

$$\mu(x) := \int_G \text{Ad}^*(g)(\tilde{\mu}(g^{-1} \cdot x)) \, d\gamma(g)$$

for all $x \in \mathcal{M}$. We claim that $\mu$ is a moment map.

Let $h \in G$ and $x \in \mathcal{M}$. Then

$$\mu(h \cdot x) = \int_G \text{Ad}^*(g)(\tilde{\mu}((h^{-1}g) \cdot x)) \, d\gamma(g)$$

$$= \int_G \text{Ad}^*(g)(\tilde{\mu}((g^{-1}h) \cdot x)) \, d\gamma(g) = \int_G \text{Ad}^*(g)(\tilde{\mu}((h^{-1}g^{-1}) \cdot x)) \, d\gamma(g)$$

$$= \int_G \text{Ad}^*(hg)(\tilde{\mu}((hh^{-1}g^{-1}) \cdot x)) \, d\gamma(g) = \int_G \text{Ad}^*(hg)(\tilde{\mu}(g^{-1} \cdot x)) \, d\gamma(g)$$

$$= \int_G \text{Ad}^*(h)\text{Ad}^*(g)(\tilde{\mu}(g^{-1} \cdot x)) \, d\gamma(g).$$

By Lemma 1.6, since $\text{Ad}^*(h)$ is a linear operator on $g^*$, we have

$$\int_G \text{Ad}^*(h)\text{Ad}^*(g)(\tilde{\mu}(g^{-1} \cdot x)) \, d\gamma(g)$$

$$= \text{Ad}^*(h)\left[ \int_G \text{Ad}^*(g)(\tilde{\mu}(g^{-1} \cdot x)) \, d\gamma(g) \right]$$

$$= \text{Ad}^*(h)\mu(x).$$

Thus $\mu$ is equivariant, which is property (1) of moment maps.

Let $\xi \in g$, $x \in \mathcal{M}$, $v \in T_x\mathcal{M}$. Let

$$\Gamma_1 : \mathbb{R} \to T_{(\tilde{\mu}(x), \xi)}\mathbb{R}$$

$$\Gamma_2 : g^* \to T_{\tilde{\mu}(x)}g^*$$

$$\Gamma_3 : g^* \to T_{\text{Ad}^*(g)(\tilde{\mu}(g^{-1} \cdot x))}g^*$$

$$\Gamma_4 : \mathbb{R} \to T_{(\tilde{\mu}(g^{-1} \cdot x), \text{Ad}(g^{-1})\xi)}\mathbb{R}$$

be the canonical linear isomorphisms. Because the canonical pairing of elements of $g^*$ with $\xi \in g$ is a linear map $g^* \to \mathbb{R}$, we can apply Lemma 3.9 to commute the taking of tangent
maps with the canonical pairing. Let $g \in G$ be any element, and denote by $\bar{\mu}$ the map $\Ad^* (g) \circ \bar{\mu} \circ \lambda^{g^{-1}}$. Then

\[(d(\bar{\mu}, \xi)) v = \Gamma_1^{-1} [T_x (\bar{\mu}, \xi)] v = \langle \Gamma_2^{-1} \circ (T_x \bar{\mu})(v), \xi \rangle \]

\[= \langle \Gamma_2^{-1} \circ T_x (\Ad^*(g) \circ \bar{\mu} \circ \lambda^{g^{-1}})(v), \xi \rangle \]

\[= \langle \Gamma_2^{-1} \circ [T_{\bar{\mu}(g^{-1} x)} \Ad^*(g)] \circ [T_{g^{-1} x} \bar{\mu}] \circ [T_x \lambda^{g^{-1}}](v), \xi \rangle \]

\[= \langle \Ad^*(g^{-1} \circ \Gamma_3^{-1} \circ [T_{g^{-1} x} \bar{\mu}] \circ [T_x \lambda^{g^{-1}}](v), \xi \rangle \]

\[= \Gamma_4^{-1} \left[ (T_{g^{-1} x} \langle \bar{\mu}, \Ad(g^{-1})\xi \rangle )(g_{x^{-1}}v) \right] \]

\[= \omega_{g^{-1} x} \left( (\Ad(g^{-1})\xi)_{\mathcal{M}}(g^{-1} x), g_{x^{-1}}v \right). \]

The last step follows from the fact that $\bar{\mu}$ satisfies property (2) of moment maps. The symbol $g_{x^{-1}}$ used above denotes the tangent map at $x$ of $\lambda^{g^{-1}}$.

Recall that for each $a \in G$ we have the map $\Psi_a : G \to G$ given by conjugation by $a$, and that $\Ad(a) := T_e \Psi_a$. Then

\[(\Ad(g^{-1})\xi)_{\mathcal{M}} := (T_e \lambda_{g^{-1} x}) (\Ad(g^{-1})\xi) \]

\[= (T_e \lambda_{g^{-1} x} \circ T_e \Psi_{g^{-1}})(\xi) \]

\[= T_e (\lambda_{g^{-1} x} \circ \Psi_{g^{-1}})(\xi). \]

Note that for each $h \in G$, we have

\[\lambda_{g^{-1} x} (\Psi_{g^{-1}}(h)) = \lambda_{g^{-1} x} (g^{-1} h g) = \lambda (g^{-1} h g, g^{-1} x) \]

\[= \lambda^{g^{-1} h g} \circ \lambda^{g^{-1}}(x) = \lambda^{g^{-1} h g} \circ \lambda^{g^{-1}}(x) = \lambda^{g^{-1} h}(x) \]

\[= \lambda^{g^{-1}} \circ \lambda_x(h). \]

Hence

\[T_e (\lambda_{g^{-1} x} \circ \Psi_{g^{-1}})(\xi) = T_e (\lambda^{g^{-1}} \circ \lambda_x)(\xi) \]

\[= (T_e \lambda^{g^{-1}} \circ T_e \lambda_x)(\xi) = g_{x^{-1}} (\xi_{\mathcal{M}}(x)). \]
Therefore
\[ \omega_{g^{-1}x} \left( (\text{Ad}(g^{-1})\xi)_M (g^{-1}x), g^{-1}xv \right) \]
\[ = \omega_{g^{-1}x} \left( g^{-1} \circ \xi_M (x), g^{-1}xv \right) \]
\[ = \omega_x \left( \xi_M (x), v \right). \]

The last step follows from the fact that $G$ acts by symplectomorphisms. Thus we have
\[ (d\langle \mu, \xi \rangle)_x v \]
\[ = \left( d\int_G \text{Ad}^*(g) (\bar{\mu}(g^{-1}x)) \, d\gamma(g), \xi \right)_x v \]
\[ = \int_G \left( d\langle \text{Ad}^*(g) (\bar{\mu}(g^{-1}x)) \xi \rangle_x \right)(v) \, d\gamma(g) \]
\[ = \int_G \omega_x \left( \xi_M (x), v \right) \, d\gamma(g) \]
\[ = \omega_x \left( \xi_M (x), v \right) \cdot \gamma(G) \]
\[ = \omega_x \left( \xi_M (x), v \right). \]

Since $x$, $\xi$, and $v$ were chosen arbitrarily, this is equivalent to the statement that $\nabla_\omega \langle \mu, \xi \rangle = \xi_M$ for all $\xi \in \mathfrak{g}$, which is precisely property (2) of moment maps.

Thus $\mu : M \to \mathfrak{g}^*$ is a moment map. QED