A LITTLE LINEAR APPROXIMATION

Problem.

(a) Estimate the value of \( \sqrt{26} \) using linear approximation.

(b) Is this an overestimate or underestimate?

(c) Approximate the size of the error in your estimate.

(d) Use the information computed above to determine an interval which you can be sure contains the exact value of \( \sqrt{26} \).

Solution.

The linear approximation:

The function which we are asked to estimate the value of is \( f(x) = \sqrt{x} \). We need to find some value \( x = A \) which is both close to \( x = 26 \) (since linear approximation works best for values of \( x \) near the center of approximation) and at which \( f(x) \) is easy to evaluate (since the whole point is to approximate \( \sqrt{26} \) with a number which we can work with easily). The clear choice is \( A = 25 \).

We have \( f'(x) = \frac{1}{2\sqrt{x}} \), and \( f(A) = f(25) = \sqrt{25} = 5 \), and \( f'(A) = f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10} \). Then the linear approximation of \( f(x) = \sqrt{x} \) centered at \( A = 25 \) is

\[ L(x) = 5 + \frac{1}{10} (x - 25), \]

as given by the formula \( L(x) = f(A) + f'(A) (x - A) \). Therefore

\[ \sqrt{26} = f(26) \approx L(26) = 5 + \frac{1}{10} (26 - 25) = 5 + \frac{1}{10} = 5.1. \]

Over or under?

To determine whether this is an overestimate or underestimate, we look to the second derivative. Since \( f'(x) = (1/2) x^{-1/2} \), we know \( f''(x) = (1/2) \cdot (-1/2) \cdot x^{-3/2} = -1/(4x^{3/2}) \). Since \( f''(x) < 0 \) for all \( x \) in the interval \((25, 26)\) where our linear approximation is taking place, we know that the graph of \( f(x) \) is concave down here, so each of its tangent lines is over the curve \( f(x) \) on this interval. Therefore \( L(26) = 5.1 \) is an overestimate of the actual value of \( \sqrt{26} \).
Approximating the size of the error:

As usual, we let \( E(x) = f(x) - L(x) \) be the error in approximating the value of \( f(x) \) using \( L(x) \). From the previous part, we know that \( f(26) < E(26) \) which means \( E(26) < 0 \). (Remember that \( E(26) = 0 \) if and only if our approximation of \( \sqrt{26} \) is actually exactly right. Our error is not zero, since \( (5.1)^2 = 26.01 \neq 26 \) This is a decent initial estimate of the error.

From the Linear Approximation Error Theorem, we know that there is a number \( s \) between 25 and 26 so that the error \( E(26) \) in our estimate is exactly

\[
E(26) = \frac{f''(s)}{2} (26 - 25)^2 = \frac{1}{2} - \frac{1}{4s^{3/2}} = -\frac{1}{8s^{3/2}}.
\]

The theorem doesn’t tell us which number \( s \) is exactly, so we want to figure out as much as possible about \( E(26) = -1/(8s^{3/2}) \) knowing only that \( 25 < s < 26 \). The size of our error is its absolute value, \( |E(26)| = |-1/(8s^{3/2})| \). Because \( s \) is positive, we know \( s^{3/2} > 0 \), so \( |1/(8s^{3/2})| = 1/(8s^{3/2}) \). We want to answer the question, “What is the worst that the error could be?”, which is equivalent to the question, “What is the largest the size of the error could be?” We want to find the maximum value of the function \( B(t) = \frac{1}{8} t^{-3/2} \) on the interval \( 25 < t < 26 \). (This is our error-bounding function.)

The derivative of \( B(t) \) is

\[
B'(t) = \frac{d}{dt} \frac{1}{8} t^{-3/2} = \left( \frac{1}{8} \right) \cdot \left( \frac{-3}{2} \right) \cdot t^{-5/2} = -\frac{3}{16} t^{-5/2}.
\]

When \( t \) is positive, \( t^{-5/2} \) is positive, so \( B'(t) \) is negative. Therefore \( B(t) \) is decreasing when \( t > 0 \). It follows that when \( 25 < t < 26 \), we have

\[
\frac{1}{8} t^{-3/2} = B(t) < B(25) = \frac{1}{8} (25)^{-3/2} = \frac{1}{8/(\sqrt{25})^3} = \frac{1}{1000} = 0.001.
\]

Therefore the size \( |E(26)| = B(s) \) of the error in our approximation satisfies

\[
|E(26)| < 0.001.
\]
An interval containing $\sqrt{26}$:

The inequality $|E(26)| < 0.001$ means that

$$-0.001 < E(26) < 0.001.$$ 

Since $E(26) = f(26) - L(26) = \sqrt{26} - 5.1$, this leads us to the following inequalities.

$$-0.001 < E(26) < 0.001,$$
$$-0.001 < \sqrt{26} - 5.1 < 0.001,$$
$$5.1 - 0.001 < \sqrt{26} < 5.1 + 0.001,$$
$$5.099 < \sqrt{26} < 5.101.$$ 

Therefore $\sqrt{26}$ is contained in the open interval $(5.099, 5.101)$. However, remember that we proved earlier that our linear approximation of $\sqrt{26}$ is an overestimate. Therefore $\sqrt{26} < 5.1$, so $\sqrt{26}$ is contained in the slightly smaller open interval $(5.099, 5.1)$.

Finding the error in linear approximation.

Assume that we have a function $f$, a linearization $L$ of $f$ centered at $A$, and that we have approximated a particular value $f(x_0)$ with $L(x_0)$. The error in this approximation is $E(x_0) = f(x_0) - L(x_0)$.

Let $I$ be the open interval with endpoints $x_0$ and $A$. Here are some steps you can follow to determine how good your linear approximation is.

1. **Compute** $f''(t)$.
   - If $f''(t) > 0$ for all $t$ in $I$, then $f$ is concave up on $I$, so $L(x_0) < f(x_0)$, so your approximation is an **under-estimate**.
   - If $f''(t) < 0$ for all $t$ in $I$, then $f$ is concave down on $I$, so $L(x_0) > f(x_0)$, so your approximation is an **over-estimate**.

2. **Write down** the following, with your particular values for $x_0$, $A$, and $I$ substituted in.

   “There is some $s$ in $I$ such that $E(x_0) = \frac{f''(s)}{2}(x_0 - A)^2$.”

3. **Simplify** your formula for $E(x_0)$, and use it to find a formula for $|E(x_0)|$. This is the **size** of your error.

4. Take your formula for $|E(x_0)|$, **substitute** the variable $t$ for $s$, and label this $B(t)$ (or whatever you want). This is your **error-bounding** function. (Note that now $|E(x_0)| = B(s)$.)
5. Find a **nice upper bound** for $B(t)$ on the interval $I$.

Most often, for us, the function $B(t)$ will be either *always increasing* or *always decreasing* on the interval $I$. (You check this by computing $B'(t)$.)

- If $B(t)$ is increasing on $I$, then plug the **right endpoint** of $I$ into $B(t)$ to get an upper bound.
- If $B(t)$ is decreasing on $I$, then plug the **left endpoint** of $I$ into $B(t)$ to get an upper bound.
- If the endpoint of $I$ that you plug into $B(t)$ gives you a **nice number**, then this is a **good upper bound**. (This will usually happen if this endpoint is $A$, which was chosen because it gives nice outputs.) If you do not get a nice number, then you should search further out past the endpoint to get a cleaner upper bound.
- For example, if $B(t) = \sqrt{t}$, which is an increasing function, then we should use the right endpoint of $I$. If the right endpoint of $I$ is 11, then we get $B(11) = \sqrt{11}$, which is not a nice upper bound. So we go further to the right, past the endpoint 11, until we reach 16. Then $B(16) = \sqrt{16} = 4$. Since $B(t)$ is always increasing, this is still an upper bound, and it is also a nice number.

6. Once you have a nice upper bound for $B(t)$ on $I$, you can conclude that $|E(x_0)| = B(s)$ is less than this upper bound too. This is your **estimate** of the size of your error. The **smaller** it is, the **better** you know your estimate is.

7. To find an interval that contains $f(x_0)$, we use the information we gathered above.

- Let $U$ be your upper bound for $|E(x_0)|$. Then

$$-U < E(x_0) < U,$$
$$-U < f(x_0) - L(x_0) < U,$$
$$L(x_0) - U < f(x_0) < L(x_0) + U.$$ 

Therefore, $f(x_0)$ is contained in the open interval $(L(x_0) - U, L(x_0) + U)$.

- If your linear approximation was an over-estimate, then replace the right endpoint $L(x_0) + U$ with $L(x_0)$ as a right endpoint for your interval instead.

- If your linear approximation was an under-estimate, then replace the left endpoint $L(x_0) - U$ with $L(x_0)$. 
