A little taste of symplectic geometry

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What is symplectic geometry?

Symplectic geometry is the study of the geometry of symplectic manifolds!
The game plan

0. Prologue: Schur–Horn theorem (original version)
1. Symplectic vector spaces
2. Symplectic manifolds
3. Hamiltonian group actions
4. Atiyah/Guillemin–Sternberg theorem
5. Epilogue: Schur–Horn theorem (symplectic version)
0. Prologue

Let $\mathcal{H}(n) = \{\text{Hermitian} \ (n \times n)\text{–matrices}\}. \ (\tilde{A}^T = A)$

Hermitian $\implies$ real diagonal entries and eigenvalues.

Put

$\vec{\lambda} = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n),$

$\mathcal{O}_{\vec{\lambda}} = \{A \in \mathcal{H}(n) \text{ with eigenvalues } \vec{\lambda}\} \text{ (isospectral set),}$

$f: \mathcal{O}_{\vec{\lambda}} \to \mathbb{R}^n, \ f(A) = \text{diagonal of } A.$

**Theorem:** [Schur-Horn, mid-1950’s]

$f(\mathcal{O}_{\vec{\lambda}})$ is a **convex** polytope in $\mathbb{R}^n$, the **convex hull** of vectors whose entries are $\lambda_1, \ldots, \lambda_n$ (in some order).

**Definition:** $C$ is **convex** if $a, b \in C \implies \overline{ab} \subset C.$

The **convex hull** of $P$ is the smallest convex set containing $P$.

A **convex polytope** is the convex hull of a finite set of points.
Example: $n = 3, \vec{\lambda} = (3, 2, 1)$.

$f(\mathcal{O}_{\vec{\lambda}})$ lives in $\mathbb{R}^3$, but is contained in the plane $x + y + z = 6$. 
1. Symplectic vector spaces

\( V = \) finite dimensional real vector space

**Definition:** An inner product on \( V \) is a map \( g : V \times V \to \mathbb{R} \) with the following properties.

- \( g \) is **bilinear**
- \( g \) is **symmetric**
- \( g \) is **positive definite**

Note: positive definite \( \implies \) nondegenerate.

**Example:** \( V = \mathbb{R}^n \), \( g = \) standard dot product
**Definition:** A **symplectic product** on $V$ is a map $\omega: V \times V \rightarrow \mathbb{R}$ with the following properties.

- $\omega$ is **bilinear**
- $\omega$ is **skew-symmetric**
- $\omega$ is **nondegenerate**

(Note that for all $v \in V$, $\omega(v, v) = 0$.)

A **symplectic vector space** is a vector space equipped with a symplectic product.

Every (finite dimensional) vector space has an inner product, but *not every vector space has a symplectic product!*
Claim: If $V$ has a symplectic product $\omega$, then $\dim V$ is even.

Proof: Let $A$ be the matrix of $\omega$ relative to some basis for $V$. Then

$$\det A = \det A^T = \det(-A) = (-1)^n \det A,$$

where $n = \dim V$. Since $\det A \neq 0$, $1 = (-1)^n$, so $n = \dim V$ is even.

QED

Example: $V = \mathbb{R}^{2n}$, $\omega = \omega_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. (standard symplectic product)

If $n = 2$:

$$\omega(\vec{x}, \vec{y}) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= x_1y_2 - x_2y_1 = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \det (\vec{x}, \vec{y})$$

$$= \text{oriented area of the parallelogram spanned by } \vec{x}, \vec{y}.$$

Thus, every even-dimensional vector space has a symplectic product, and in fact, up to a change of coordinates, every symplectic product looks like this one!
The **gradient** of $f: \mathbb{R}^n \to \mathbb{R}$ is the vector field

$$\nabla f := \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

**Coordinate-free definition:** $\nabla f$ is the unique vector field such that $\forall p \in \mathbb{R}^n, \vec{v} \in \mathbb{R}^n$,

$$\left( D_{\vec{v}} f \right)(x) = \nabla f(x) \cdot \vec{v}.$$  

($D_{\vec{v}} f = \text{directional derivative}$ of $f$ in the direction $\vec{v}$.)

The **symplectic gradient** of $f$ is the unique vector field $\nabla \omega f$ such that $\forall p \in \mathbb{R}^n, \vec{v} \in \mathbb{R}^n$,

$$\left( D_{\vec{v}} f \right)(x) = \omega \left( \nabla \omega f(x), \vec{v} \right).$$

(The uniqueness follows from nondegeneracy.)
Example: \( V = \mathbb{R}^2 \), \( \omega = \omega_0 \) = standard symplectic form.

\[
\nabla_\omega f = \left( -\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x} \right)
\]

Let \( f(x, y) = x^2 + y^2 \). Then

\[
\nabla f = (2x, 2y) \quad \text{and} \quad \nabla_\omega f = (-2y, 2x).
\]

\( \nabla f \) is perpendicular to level curves of \( f \), and points to increasing values of \( f \).

\( \nabla_\omega f \) is tangent to level curves of \( f \), and points to constant values of \( f \).

\[
(\mathcal{D}_{\nabla_\omega f(p)} f)(p) = \omega (\nabla_\omega f(p), \nabla_\omega f(p)) = 0.
\]

\( f \rightsquigarrow \) energy function
\( \nabla f \rightsquigarrow \) points to increasing energy
\( \nabla_\omega f \rightsquigarrow \) points to stable energy

*Symplectic geometry is the natural setting for studying classical mechanics!*
A game we can play: Find the Hamiltonian!

Usual version
Given a vector field $X$ on $\mathbb{R}^n$, find a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ such that
$$\nabla f = X.$$ 

Symplectic version
Given a vector field $X$ on $V$, find a function $f: V \rightarrow \mathbb{R}$ such that
$$\nabla_\omega f = X.$$ 

Classical mechanics interpretation
The vector field represents a system of moving particles (Hamiltonian system). We want to find an energy function (Hamiltonian) for this system.

We are basically trying to solve Hamilton’s equations.
2. Symplectic manifolds

**Definition:** A smooth manifold $M$ consists of “patches” (open subsets of some $\mathbb{R}^n$) smoothly knit together.

(Think of smooth surfaces in $\mathbb{R}^3$, like a sphere or torus.)

Each point $p \in M$ has a tangent space $T_p M$ attached.

A **Riemannian metric** on $M$ is a smoothly varying collection

$$g = \{g_p : T_p M \times T_p M \to \mathbb{R} \mid p \in M\}$$

of inner products.

A **symplectic form** on $M$ is a smoothly varying collection

$$\omega = \{\omega_p : T_p M \times T_p M \to \mathbb{R} \mid p \in M\}$$

of symplectic products, such that $d\omega = 0$. 
Every manifold has a Riemannian metric (partition of unity), but not every manifold admits a symplectic form!

Being even-dimensional and orientable is necessary but not sufficient!

Example: \( M = \) orientable surface in \( \mathbb{R}^3 \), \( \omega(\vec{u}, \vec{v}) = \) oriented area of parallelogram spanned \( \vec{u} \) and \( \vec{v} \).

Fact: Locally, every symplectic manifold looks like \( (\mathbb{R}^{2n}, \omega_0) \). (Darboux’s theorem)

(No local invariants in symplectic geometry, like curvature.)

Can define gradients just like before.

\[
\forall p \in M, \vec{v} \in T_pM, \quad df_p(\vec{v}) = \omega_p(\nabla \omega f(p), \vec{v})
\]

differentiable function \( f: M \to \mathbb{R} \) \( \leadsto \) tangent vector field \( \nabla f, \nabla \omega f \)
Example: \( M = S^2 := \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \} \),
g = \text{dot product}, \ \omega = \text{oriented area}

\( f: S^2 \to \mathbb{R}, (x, y, z) \mapsto z \), (height function)

\( \nabla f \) points longitudinally, \( \nabla_\omega f \) points latitudinally

As before:

\( \nabla f \) points to increasing values of \( f \),
\( \nabla_\omega f \) points to constant values of \( f \).

We can still play “Find the Hamiltonian!”. Given a tangent vector field \( X \) on \( M \), can we find a function \( f: M \to \mathbb{R} \) such that

- \( \nabla f = X \)?
- \( \nabla_\omega f = X \)?
3. Hamiltonian group actions

**Definition:** A **Lie group** is a group $G$ with a compatible structure of a smooth manifold.

A **smooth action** of $G$ on a smooth manifold $M$ is a “smooth” group homomorphism $\mathcal{A}: G \to \text{Diff}(M)$.

\[ \text{Diff}(M) = \text{diffeomorphisms } M \to M. \]

The **Lie algebra** $\mathfrak{g}$ of $G$ is the tangent space at the identity element $1$ of $G$.

\[ \mathfrak{g} := T_1G \]

$\mathfrak{g}$ is a vector space, and more. (Lie bracket)
**Example:** Some Lie groups.

(i) \((V, +)\).
   Lie algebra \(\cong V\).

(ii) \(S^1 := \{z \in \mathbb{C} \mid |z| = 1\}\) under multiplication.
   Lie algebra \(= i\mathbb{R}\).

(iii) \(T = S^1 \times \ldots \times S^1\), a **torus**.
   Lie algebra \(= i\mathbb{R} \oplus \ldots \oplus i\mathbb{R}\).

(iv) Matrix Lie groups under matrix multiplication, such as
   \(GL(n; \mathbb{R}), SL(n; \mathbb{R}), O(n; \mathbb{R}), SO(n; \mathbb{R}), U(n)\), etc.
   Their Lie algebras are certain matrix vector spaces.

**Example:** A smooth group action. (Rotating the plane.)

Let \(G = S^1\), \(g = i\mathbb{R}\), \(M = \mathbb{R}^2\), and \(A: S^1 \to \text{Diff}(\mathbb{R}^2)\) be defined by

\[
A(e^{i\theta}) \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]
$1 \in G$ acts as the identity map $M \to M$.

**infinitesimal change** in $G$ at $1 \leadsto$ **infinitesimal change** at each $p \in M$.

An infinitesimal change at $1 \in G$ is some $\xi \in g$. An infinitesimal change at each $p \in M$ is a vector field.

$$g \to \text{Vec}(M), \quad \xi \mapsto \xi_M$$

$\xi_M$ is the **fundamental vector field** on $M$ induced by $\xi$.

$$\xi_M(p) := \frac{d}{dt} A(\exp(t \xi)) p \bigg|_{t=0}$$

In the example of rotating the plane, if $\xi = it \in i\mathbb{R} = g$, then

$$\xi_M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -ty \\ tx \end{pmatrix}.$$
We can play “Find the Hamiltonian!” with the fundamental vector fields.

If we can win this game for every $\xi_M$, then we can form $\phi: g \to C^\infty(M)$ such that for every $\xi \in g$

$$\nabla_\omega [\phi(\xi)] = \xi_M.$$

($C^\infty(M) = \{\text{smooth functions } M \to \mathbb{R}\}.$)

Take “dual”, and define $\Phi: M \to g^*$ by

$$\Phi(p)\xi = \phi(\xi)(p)$$

for all $p \in M$, $\xi \in g$.

If $\Phi$ is also $G$–equivariant then $\Phi$ is a moment map for $\mathcal{A}: G \to \text{Diff}(M)$.

$\mathcal{A}$ is a Hamiltonian action of $G$ on $M$ if there is a moment map $\Phi$ for the action.

$$G \curvearrowright M \xrightarrow{\Phi} g^*.$$
Examples:

(i) Rotating the plane. \( G = S^1, g = i\mathbb{R}, M = \mathbb{R}^2 \).
\( \Phi : M \to g^* \) is

\[
\Phi \left( \begin{array}{c} x \\ y \end{array} \right) (it) = \left( \frac{1}{2}(x^2 + y^2) \right) t
\]

Note that \( \nabla_\omega \) of this function on \( \mathbb{R}^2 \) is \( \left( \begin{array}{c} -ty \\ tx \end{array} \right) = (it)_M \left( \begin{array}{c} x \\ y \end{array} \right) \).

(ii) \( M = \mathbb{R}^6 \) with coordinates \( \vec{x}, \vec{y} \in \mathbb{R}^3 \). (\( \vec{x} \) is position, \( \vec{y} \) is momentum).
\( G = \mathbb{R}^3 \) acting on \( M \) by translating the position vector.
Then \( g = \mathbb{R}^3 \cong g^* \), and \( \Phi : M \to g^* \) is

\[
\Phi(\vec{x}, \vec{y}) \vec{a} = \vec{y} \cdot \vec{a}.
\]

\( \Phi = \text{linear momentum} \).

(iii) \( M = \text{cotangent bundle of} \ \mathbb{R}^3 \) with coordinates \( \vec{x}, \vec{y} \in \mathbb{R}^3 \). (\( \vec{x} \) is still position, \( \vec{y} \) is still momentum).
\( G = \text{SO}(3) \) acting on \( M \) by “rotation”. Then \( g^* \cong \mathbb{R}^3 \), and \( \Phi : M \to g^* \) is

\[
\Phi(\vec{x}, \vec{y}) \vec{a} = (\vec{x} \times \vec{y}) \cdot \vec{a}.
\]

\( \Phi = \text{angular momentum} \).
4. Atiyah/Guillemin–Sternberg Theorem


G–S proof: “simple and elegant”

A proof: “even a bit more simple and elegant”

**Theorem:**

\[(M, \omega) = \text{compact and connected symplectic manifold, T = a torus,} \]

\[\mathcal{A} = \text{Hamiltonian action of T on M with moment map } \Phi: M \to t^*.\]

Then \(\Phi(M)\) is a **convex polytope** in \(t^*\), the **convex hull** of \(\Phi(M^T)\), where

\[M^T := \{p \in M | \mathcal{A}(t)p = p \text{ for all } t \in T\}.\]
5. Epilogue

**Symplectic interpretation of Schur-Horn theorem:**

Noticed by Bertram Kostant in the early 1970’s, then generalized by A/G–S.

- \( U(n) = \{ A \mid \bar{A}^T = A^{-1} \} \) is a Lie group. Acts on \( \mathcal{H}(n) \) by conjugation. \( \mathcal{H}(n) \cong u(n)^* \)

- \( T \) = diagonal matrices in \( U(n) \) is an \( n \)–torus. Can identify \( t^* \) with \( \mathbb{R}^n \).

- \( O_{\vec{\lambda}} \) = isospectral set for \( \vec{\lambda} \) is a symplectic manifold. (coadjoint orbit, Kirillov–Kostant–Souriau form)

- Conjugation preserves eigenvalues, so \( T \ltimes O_{\vec{\lambda}} \).

- \( f: O_{\vec{\lambda}} \to \mathbb{R}^n, f(A) = \text{diagonal of } A \), is a moment map.

- \( (O_{\vec{\lambda}})^T = \text{diagonal matrices in } O_{\vec{\lambda}} = \text{diagonal matrices with entries } \lambda_1, \ldots, \lambda_n \) in some order.

**A/G–S theorem** \( \implies f(O_{\vec{\lambda}}) \) is a convex polytope, the convex hull of \( f \left( (O_{\vec{\lambda}})^T \right) \).

This is exactly the **S–H theorem**!
Symplectic stuff is cool! But you don’t have to take my word for it!

Coming Spring 2008:

Tara Holm’s NEW epic
MATH 758: Symplectic Geometry
THE END

Thank you for listening.