Math 4370 Final Exam

Handed out May 4th 2010

Due May 13th 2010

Problem 1. Recall that an element $f \in k[x_1, \ldots, x_n]$ is *homogeneous of degree* i if all of the terms of f have the same total degree i. An ideal $I \subseteq k[x_1, \ldots, x_n]$ is *homogeneous* if I can be generated by a set of homogeneous elements. Every element $f \in k[x_1, \ldots, x_n]$ can be written uniquely as $f = \sum_{i \in \mathbb{N}} f_i$ where only finitely many of the f_i are nonzero, and the f_i are homogeneous of degree i (you may assume this). The f_i are called the *homogeneous components* of f. If I is a homogeneous ideal, we denote those elements of total degree i in I by I_i.

(a) Show that I is homogeneous if and only of I satisfies the following condition: $f \in I$ if and only if every homogeneous component of f is in I.

(b) Find a vector space basis of the set of homogeneous polynomials of degree m in $k[x_1, \ldots, x_n]$ (it may help to do the case $n = 2, 3$ and for several m). Find a formula for its dimension.

(c) Use part (a) to show that if I is a homogeneous ideal and f is a homogeneous element of $k[x_1, \ldots, x_n]$, then $(I : f)$ is also homogeneous.

(d) Show that if $>$ is any term order, then a homogeneous ideal I has a homogeneous Gröbner basis.

(e) Let $\varphi \colon k[x_1, \ldots, x_n] \longrightarrow k[y_1, \ldots, y_m]$ be a ring homomorphism such that $\varphi(x_i)$ is homogeneous for all i, where each of the y_i have degree 1. Give the variables x_i the same degree as $\varphi(x_i)$, so that φ now preserves degrees and is what is known as a *graded* ring homomorphism. Use part (d) to show that $\ker \varphi = \{f \in k[x_1, \ldots, x_n] \mid \varphi(f) = 0\}$ is a homogeneous ideal of $k[x_1, \ldots, x_n]$.

Problem 2. Let $>$ denote the grevlex order on monomials of $k[x_1, \ldots, x_n]$. That is, $x^\alpha > x^\beta$ if and only if either $|\alpha| > |\beta|$ or the rightmost nonzero entry in $\alpha - \beta$ is negative. In what follows, $\mathrm{LT}_>(J)$ denotes the ideal of lead terms of the ideal J with respect to the term order $>$.

(a) Show that for every homogeneous $f \in k[x_1, \ldots, x_n]$, $x_n \mid \mathrm{LT}_>(f)$ if and only if $x_n \mid f$.

(b) Show that for each $1 \le i \le n$, and for every homogeneous ideal I, one has

$$\langle \mathrm{LT}_>(\langle I, x_n, \ldots, x_i \rangle) \rangle = \langle \mathrm{LT}_>(I), x_n, \ldots, x_i \rangle.$$

Show by example that equality does not hold if $>$ is the lex ordering.

(c) Using the notation from problem 1, suppose that I is a homogeneous ideal and that $x_n, \ldots, x_{i+1} \in I$. Show that for all $m \ge 0$, one has an equivalence

$$(I : x_i)_m = I_m \Leftrightarrow (\mathrm{LT}_>(I) : x_i)_m = \mathrm{LT}_>(I)_m.$$

Again, produce an example that shows this equivalence does not hold for the lex ordering.

Problem 3. Let $I \subseteq k[x_1, \ldots, x_n]$ be a monomial ideal.

(a) Show that for any $f \in k[x_1, \ldots, x_n]$ and any monomial $m \in I$, one has $(I : f) = (I : f + m)$.

(b) Let $f \in k[x_1, \ldots, x_n]$. Show that if $(I : f)$ is prime, then $(I : f)$ is monomial, and is equal to $(I : m)$ for some monomial m. Show that if I is a squarefree monomial ideal, then m can be taken to be squarefree as well. (Hint: Use part (a) to show that we can assume that no term of f is in I. Now induct on the number of terms of f while making use of some term order $>$ on the monomials $k[x_1, \ldots, x_n]$.)

(c) Show that $(I : f)$ need not be a monomial ideal if we do not assume that $(I : f)$ is prime.

(d) Conclude that every squarefree monomial ideal is the intersection of ideals of the form $\langle x_{i_1}, \dots, x_{i_l} \rangle$ for $1 \le i_1 < \cdots < i_l \le n$. (Remember what we know about squarefree monomial ideals from an old homework!)

(e) What does the previous problem say about varieties defined by squarefree monomial ideals?

(f) Express the ideals $\langle x_1x_2, x_3x_4 \rangle$ and $\langle x_1x_3, x_1x_4, x_2x_3, x_2x_4 \rangle$ in the way described in part (d). Do you see a connection between these two examples? (Hint: Start computing $(I : m)$ for various (squarefree) monomials m to find all the colons that have the form from part (d) (Macaulay2 is good at this!). Since there are only finitely many squarefree monomials, you can find all the components this way.)

Problem 4. Let $G, H \subseteq \mathrm{GL}(n, k)$ be finite matrix groups, where k is a field with $\mathbb{Q} \subset k$, and let G act on $k[x_1, \dots, x_n]$ in the usual way.

(a) If $G \subseteq H$, show that $k[x_1, \dots, x_n]^H \subseteq k[x_1, \dots, x_n]^G$.

(b) Show that if $f_1, \dots, f_l$ are the algebra generators of $k[x_1, \dots, x_n]^G$ coming from Noether's theorem, then the ideal of relations among the $\{f_i\}$ is a homogeneous ideal. (Use the method we used in class to think of the ideal of relations among the $\{f_i\}$ as the kernel of a ring homomorphism satisfying the conditions from problem 1.(f). Your proof should show that $\ker \varphi$ is homogeneous for *any* homogeneous generating set of $k[x_1, \dots, x_n]^G$.)

(c) Let

$$G = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\rangle \subseteq \mathrm{GL}(2, k).$$

Show that $G \cong C_6$, the cyclic group of order 6.

(d) Compute the generators of $k[x_1, \dots, x_n]^G$ using the Noether's theorem regarding the Reynolds operator (Theorem 7.3.5). See the code online to help you do this in Macaulay2.

(e) Compute a set of generators for the ideal of relations among the generators from part (d) using a computer algebra package.

Problem 5. Let $\triangle ABC$ be a triangle in the plane. Look up the definitions of circumcenter, orthocenter, and centroid of a triangle. Use the methods we developed in class to prove that the circumcenter, orthocenter and centroid of $\triangle ABC$ are collinear. Be sure to include a printout of the code you used to complete this problem in a computer algebra package.

Problem 6. The purpose of this exercise is to show that if k is any field which is not algebraically closed, then any variety $V \subseteq k^n$ can be defined by a single equation (note that you have done this for k any ordered field on your homework, but here we extend it for any field that is not algebraically closed).

(a) If $f = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ is a polynomial of degree n in x, define the homogenization of f with respect to y, denoted f^h, to be the homogeneous polynomial $f = a_0x^n + a_1x^{n-1}y + \cdots + a_{n-1}xy^{n-1} + a_ny^n$. Show that f has a root in k if and only if there is an $(a, b) \in k^2$ such that $(a, b) \ne (0, 0)$ and $f^h(a, b) = 0$. (Hint: Show that $f^h(a, b) = b^n f^h(a/b, 1)$ when $b \ne 0$.)

(b) If k is not algebraically closed, show that there exists $f \in k[x, y]$ such that $V(f) = \{(0, 0)\}$.

(c) If k is not algebraically closed, show that for each integer $s > 0$, there exists $f \in k[x_1, \dots, x_s]$ such that $V(f) = \{(0, \dots, 0)\}$. (Use induction, plus part (b) above).

(d) If $W = V(g_1, \dots, g_s)$ is any variety in k^n with k not algebraically closed, show that W can be defined by a single equation.