MINIMAL INTERSECTIONS AND VANISHING (CO)HOMOLOGY

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ABSTRACT. We introduce a class of local Noetherian rings, which we call minimal intersections, and show that over such rings there exist classes of modules for which the derived functors Ext and Tor vanish non-trivially. This generalizes a well-known phenomenon of non-trivial vanishing of Ext and Tor for modules over complete intersections of codimension at least two.

1. Introduction

Let $R$ be a commutative local Noetherian ring, and $M$ and $N$ finitely generated $R$-modules. In many cases the vanishing of all higher Ext and Tor can only occur in a trivial way. For instance, in [9, 11, 12, 20] it is shown that over hypersurfaces (which are codimension one complete intersections), Golod rings and Gorenstein rings of low codimension, the vanishing of all higher $\text{Tor}^R_i(M, N)$ or $\text{Ext}^R_i(M, N)$ implies that either $M$ has finite projective dimension, or $N$ has finite projective dimension (or finite injective dimension for Ext vanishing if $R$ is not Gorenstein). This raises a question of the rarity of non-trivial vanishing of all higher homology and cohomology over local rings.

The most well-known class of local rings over which the vanishing of all higher Ext and Tor occurs non-trivially is that of complete intersections of codimension at least two (see, for example, [14, Theorem 3.1], and [4]). In this paper we isolate a property of complete intersections which enables non-trivial vanishing, and consider, more generally, local Noetherian rings having this property:

Definition. Let $R = Q/I$ with $Q$ a regular local ring and $I$ an ideal in the square of the maximal ideal of $Q$. We say that $R$ is a minimal intersection (with respect to $Q$) if $I$ is the sum of two non-zero ideals $I_1$ and $I_2$ of $Q$ such that $I_1 \cap I_2 = I_1 I_2$.

We prove that there exist classes of modules over a minimal intersection demonstrating non-trivial vanishing of all higher homology and cohomology. That is, if $R$ is a minimal intersection then there exist classes of finitely generated $R$-modules $M$ and $N$ of infinite projective dimension over $R$, and (not necessarily finitely generated) $R$-modules $L$ of infinite injective dimension over $R$, such that $\text{Tor}^R_i(M, N) = 0$ for all $i \gg 0$, and $\text{Ext}^R_i(M, L) = 0$ for all $i \gg 0$. Outside of the special case of both $Q/I_1$ and $Q/I_2$ having only finitely many non-isomorphic indecomposable syzygy (or cosyzygy) modules, these classes of modules exhibiting non-trivial vanishing are

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quite large. If we further assume that $R$ is Cohen-Macaulay, then they consist of finitely generated maximal Cohen-Macaulay modules.

Minimal intersections are a generalization of complete intersections of codimension two or greater. For if $I$ is generated by a regular sequence $f_1, \ldots, f_c$ with $c \geq 2$, then for $1 \leq r \leq c$ we have $(f_1, \ldots, f_r) \cap (f_{r+1}, \ldots, f_c) = (f_1, \ldots, f_r)(f_{r+1}, \ldots, f_c)$.

In Section 2 we give general results for Ext and Tor that are needed in subsequent sections. We prove in Section 3 properties of minimal intersections which are also needed in subsequent sections. For instance, we show that the Cohen-Macaulay and Gorenstein properties are preserved after minimal intersection. Section 4 is the main part of the paper. We discuss how non-trivial vanishing of Ext and Tor can occur over complete intersections of codimension at least two, and then prove the same result assuming only that $R$ is a minimal intersection. We also show that such non-trivial vanishing over Cohen-Macaulay and Gorenstein minimal intersections (respectively) behaves similarly to that over complete intersections. The final Section 5 gives several examples, and a sufficient condition for detecting modules in the classes exhibiting non-trivial vanishing. This sufficient condition looks at the form of the free resolution of the module, and is aptly implemented on the computer. We do this using the computer algebra package Macaulay 2.

2. General Results on Ext and Tor

In this section we give some preliminary results involving Ext and Tor. We make major use of the following standard result (see, for example, [19, 11.51]).

2.1. Suppose that $A$ is a commutative ring, $J$ an ideal of $A$, and set $B = A/J$.

(1) If $X$ is an $A$-module such that $\operatorname{Tor}_i^A(X, B) = 0$ for all $i \geq 1$, then for any $B$-module $Y$ we have

$$\operatorname{Tor}_i^A(X, Y) \cong \operatorname{Tor}_i^B(X \otimes_A B, Y) \quad \text{for all } i,$$

and

$$\operatorname{Ext}_i^A(X, Y) \cong \operatorname{Ext}_i^B(X \otimes_A B, Y) \quad \text{for all } i.$$

(2) If $Y$ is an $A$-module such that $\operatorname{Ext}_i^A(B, Y) = 0$ for all $i \geq 1$, then for any $B$-module $X$ we have

$$\operatorname{Ext}_i^A(X, Y) \cong \operatorname{Ext}_i^B(X, \operatorname{Hom}_A(B, Y)) \quad \text{for all } i.$$

The following formula from [12, 2.2] is instrumental to the proofs in the subsequent sections.

2.2. Let $X$ and $Y$ be finitely generated modules over a local Noetherian ring $A$ with $\operatorname{pd}_A X < \infty$. Then

$$\sup\{i \mid \operatorname{Tor}_i^A(X, Y) \neq 0\} = \sup\{\operatorname{depth}_{A_p} A_p - \operatorname{depth}_{A_p} X_p - \operatorname{depth}_{A_p} Y_p\},$$

where the second sup is taken over all $p \in \operatorname{Spec} A$.

Proposition 2.3. Let $A$ be a Gorenstein local ring, $B = A/J$ such that $\operatorname{pd}_A B < \infty$, and $X$ be a maximal Cohen-Macaulay $A$-module. Then we have

$$\operatorname{Hom}_A(X, A) \otimes_A B \cong \operatorname{Hom}_B(X \otimes_A B, B).$$
Proof. Hom-tensor adjointness gives \( \text{Hom}_B(X \otimes_A B, B) \cong \text{Hom}_A(X, B) \), therefore it suffices to exhibit an isomorphism \( \text{Hom}_A(X, A) \otimes_A B \cong \text{Hom}_A(X, B) \). This isomorphism is easily seen when \( X \) is a free \( A \)-module. In general, let \( G \to F \to X \to 0 \) be an \( A \)-free presentation of \( X \). On the one hand we apply \( \text{Hom}_A(-, B) \), and on the other hand we apply \( \text{Hom}_A(-, A) \) first then \(- \otimes_A B\). The result is a commutative diagram:

\[
\begin{array}{c}
0 & \to & \text{Hom}_A(X, B) & \to & \text{Hom}_A(F, B) & \to & \text{Hom}_A(G, B) \\
& & \downarrow \cong & & \downarrow \cong & & \\
0 & \to & \text{Hom}_A(X, A) \otimes_A B & \to & \text{Hom}_A(F, A) \otimes_A B & \to & \text{Hom}_A(G, A) \otimes_A B \\
\end{array}
\]

We just need to know that the bottom row is exact to establish the proposition. For this, consider the short exact sequences \( 0 \to \Omega \to F \to X \to 0 \), and \( 0 \to \Omega' \to G \to \Omega \to 0 \). Applying \( \text{Hom}_A(-, A) \), and using the fact that \( \text{Ext}^i_A(X, A) = \text{Ext}^i_A(\Omega, A) = 0 \) (since \( X \) and \( \Omega \) are maximal Cohen-Macaulay \( A \)-modules), we get the short exact sequences \( 0 \to \text{Hom}_A(X, A) \to \text{Hom}_A(F, A) \to \text{Hom}_A(\Omega, A) \to 0 \) and \( 0 \to \text{Hom}_A(\Omega, A) \to \text{Hom}_A(G, A) \to \text{Hom}_A(\Omega', A) \to 0 \). Now applying \(- \otimes_A B\) we obtain

\[
\text{Tor}^1_A(\text{Hom}_A(\Omega, A), B) \to \text{Hom}_A(X, A) \otimes_A B \to \\
\text{Hom}_A(F, A) \otimes_A B \to \text{Hom}_A(\Omega, A) \otimes_A B \to 0
\]

and

\[
\text{Tor}^1_A(\text{Hom}_A(\Omega', A), B) \to \text{Hom}_A(\Omega, A) \otimes_A B \to \\
\text{Hom}_A(G, A) \otimes_A B \to \text{Hom}_A(\Omega', A) \otimes_A B \to 0.
\]

Since \( \text{Hom}_A(\Omega, A) \) and \( \text{Hom}_A(\Omega', A) \) are maximal Cohen-Macaulay \( A \)-modules and \( \text{pd}_A B < \infty \), 2.2 shows that \( \text{Tor}^1_A(\text{Hom}_A(\Omega, A), B) = \text{Tor}^1_A(\text{Hom}_A(\Omega', A), B) = 0 \) for all \( i > 0 \), and so we have short exact sequences

\[
0 \to \text{Hom}_A(X, A) \otimes_A B \to \text{Hom}_A(F, A) \otimes_A B \to \text{Hom}_A(\Omega, A) \otimes_A B \to 0
\]

and

\[
0 \to \text{Hom}_A(\Omega, A) \otimes_A B \to \text{Hom}_A(G, A) \otimes_A B \to \text{Hom}_A(\Omega', A) \otimes_A B \to 0.
\]

Splicing these together, we see that the bottom row of the diagram above is exact. \( \square \)

Proposition 2.4. Assume that \( A \) is a Cohen-Macaulay local ring with canonical module \( \omega \). Let \((-)^\vee\) denote the dual \( \text{Hom}_A(-, \omega) \). Let \( X \) and \( Y \) be finitely generated \( A \)-modules, with \( Y \) maximal Cohen-Macaulay. Then \( \text{Ext}^i_A(X, Y) = 0 \) for all \( i \gg 0 \) if and only if \( \text{Tor}^i_A(X, Y^\vee) = 0 \) for all \( i \gg 0 \).

If \( X \) is moreover maximal Cohen-Macaulay, then the following are equivalent:

1. \( \text{Ext}^i_A(X, Y) = 0 \) for all \( i \geq 1 \);
2. \( \text{Tor}^i_A(X, Y^\vee) = 0 \) for all \( i \geq 1 \), and \( X \otimes_A Y^\vee \) is maximal Cohen-Macaulay.

Proof. That (1) and (2) are equivalent is proven in [16, 2.7]. To prove the “if” direction of the first statement, choose a sufficiently high syzygy module \( \Omega^n(X) \) of \( X \) such that \( \text{Tor}^i_A(\Omega^n(X), Y^\vee) = 0 \) for all \( i \geq 1 \). This yields short exact sequences

\[
0 \to \Omega^{n+i}(X) \otimes_A Y^\vee \to F_{n+i-1} \otimes_A Y^\vee \to \Omega^{n+i-1}(X) \otimes_A Y^\vee \to 0
\]
for all \( i \geq 1 \), derived from a minimal free resolution

\[
\cdots \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow X \rightarrow 0
\]

of \( X \). Since \( F_{n+i-1} \otimes_A Y^\vee \) are maximal Cohen-Macaulay for all \( i \), counting depths along these short exact sequences shows that \( \Omega^{n+i}(X) \otimes_A Y^\vee \) are maximal Cohen-Macaulay for all \( i \geq d = \dim A \). Thus we have \( \Tor^A_i(\Omega^{n+d}(X), Y^\vee) = 0 \) for all \( i \geq 1 \), and \( \Ext^i_A(\Omega^{n+d}(X), Y) = 0 \) for all \( i \geq 0 \).

The “only if” direction of the first statement also uses the second part of the theorem, and is easier. \( \square \)

3. Minimal Intersections

Throughout this section we assume that \( Q \) is a regular local ring, and \( R = Q/(I_1 + I_2) \) with \( I_1 \) and \( I_2 \) nonzero ideals contained in the square of the maximal ideal of \( Q \), and we set \( R_1 = Q/I_1 \) and \( R_2 = Q/I_2 \). We start with some basic facts.

3.1. The ring \( R \) is a minimal intersection if and only if \( \Tor^Q_i(R_1, R_2) = 0 \) for all \( i \geq 1 \). Indeed, it is standard that \( \Tor^Q_1(R_1, R_2) = 0 \) if and only if \( I_1 \cap I_2 = I_1I_2 \) (see for example [19]). The statement follows now from rigidity of Tor for regular local rings [1, 18].

Lemma 3.2. Assume that \( R \) is a minimal intersection. Then

1. \( \pd_Q R = \pd_Q R_1 + \pd_Q R_2 \);
2. \( \depth_Q Q + \depth_Q R = \depth_Q R_1 + \depth_Q R_2 \).

Proof. The first statement follows by noting that if \( F \) and \( G \) are minimal free resolutions of \( R_1 \) and \( R_2 \) over \( Q \), then \( F \otimes_Q G \) is a minimal free resolution of \( R \cong R_1 \otimes_Q R_2 \) over \( Q \), by 3.1, and this resolution is of length \( \pd_Q R_1 + \pd_Q R_2 \).

Statement (2) follows from Statement (1) and the Auslander-Buchsbaum formula:

\[
\depth_Q Q + \depth_Q R = \depth_Q Q + (\depth_Q Q - \pd_Q R) \\
= \depth_Q Q + (\depth_Q Q - (\pd_Q R_1 + \pd_Q R_2)) \\
=(\depth_Q Q - \pd_Q R_1) + (\depth_Q Q - \pd_Q R_2) \\
= \depth_Q R_1 + \depth_Q R_2.
\]

The following result discusses Cohen-Macaulay and Gorenstein minimal intersections.

Proposition 3.3. With the notation above, we have:

1. \( R \) is Cohen-Macaulay if and only if both \( R_1 \) and \( R_2 \) are Cohen-Macaulay.

When this is the case, height(\( I_1 + I_2 \)) = height \( I_1 \) + height \( I_2 \).

2. \( R \) is Gorenstein if and only if both \( R_1 \) and \( R_2 \) are Gorenstein.

3. \( R \) is a complete intersection if and only if both \( R_1 \) and \( R_2 \) are complete intersections.

Proof. The “if” direction of (1) is given in [8, Lemma 1.10]. For the “only if” direction we use the the Intersection Theorem of Peskine-Szpiro and Roberts, which implies the inequality

\[
\dim Q + \dim R \geq \dim R_1 + \dim R_2.
\]
Therefore by Lemma 3.2(2)
\[ \text{depth } R_1 + \text{depth } R_2 = \text{depth}_Q Q + \text{depth}_Q R \]
\[ = \dim Q + \dim R \]
\[ \geq \dim R_1 + \text{depth } R_2. \]
Thus depth $R_1 \geq \dim R_1$ and so $R_1$ is Cohen-Macaulay. By symmetry, so is $R_2$.

To prove the second statement of (1), the statement of Lemma 3.2(2) gives
\[
\text{height } I = \dim Q - \dim R
\]
\[= \text{depth}_Q Q - \text{depth}_Q R \]
\[= \text{depth}_Q Q - (\text{depth}_Q R_1 + \text{depth}_Q R_2 - \text{depth}_Q Q) \]
\[= (\text{depth}_Q Q - \text{depth}_Q R_1) + (\text{depth}_Q Q - \text{depth}_Q R_2) \]
\[= (\dim Q - \dim R_1) + (\dim Q - \dim R_2) \]
\[= \text{height } I_1 + \text{height } I_2. \]

To prove (2) it suffices by (1) to show only that
\[ \text{Ext}_Q^{\text{pd}_Q R_1 + \text{pd}_Q R_2} (R, Q) \cong R \]
if and only if both
\[ \text{Ext}_Q^{\text{pd}_Q R_1} (R_1, Q) \cong R_1 \quad \text{and} \quad \text{Ext}_Q^{\text{pd}_Q R_2} (R_2, Q) \cong R_2, \]
assuming $R$ is Cohen-Macaulay. Let $F$ and $G$ be (deleted) minimal $Q$-free resolutions of $R_1$ and $R_2$, respectively. By the vanishing of $\text{Tor}_i^Q (R_1, R_2) = 0$ for all $i \geq 1$, a minimal $Q$-free resolution of $R$ is given by $F \otimes_Q G$. Let $(-)^*$ denote the dual $Q$-module. Since $R_1$ and $R_2$ are Cohen-Macaulay, both $F^*$ and $G^*$ are complexes with homology $\text{Ext}_Q^{\text{pd}_Q R_1} (R_1, Q)$ and $\text{Ext}_Q^{\text{pd}_Q R_2} (R_2, Q)$, respectively, and since $R$ is Cohen-Macaulay, $(F \otimes_Q G)^* \cong (F \otimes_Q G^* \otimes_Q G)^* \cong (F^* \otimes_Q G^*)^*$ it follows that
\[
\text{Ext}_Q^{\text{pd}_Q R_1 + \text{pd}_Q R_2} (R, Q) \cong (\text{Ext}_Q^{\text{pd}_Q R_1} (R_1, Q) \otimes_Q \text{Ext}_Q^{\text{pd}_Q R_2} (R_2, Q)).
\]
Now it is clear that $R$ is Gorenstein if $R_1$ and $R_2$ are Gorenstein. For the converse, one concludes that if $\text{Ext}_Q^{\text{pd}_2 R_1 + \text{pd}_2 R_2} (R, Q) \cong R$, then $\text{Ext}_Q^{\text{pd}_2 R_1} (R_1, Q) \cong Q/I'_1$ and $\text{Ext}_Q^{\text{pd}_2 R_2} (R_2, Q) \cong Q/I'_2$ for ideals $I'_1$ and $I'_2$ of $Q$ satisfying $I_1 \subseteq I'_1$ and $I_2 \subseteq I'_2$. Dualizing $F^*$ and $G^*$ back to $F$ and $G$ shows the reverse inclusions of ideals, yielding $I_1 = I'_1$ and $I_2 = I'_2$.

Statement (3) follows easily from (1) and the fact that $\mu_Q (I) = \mu_Q (I_1) + \mu_Q (I_2)$, where $\mu_Q (J)$ denotes the minimal number of generators of an ideal $J$ of $Q$. \hfill \Box

**Theorem 3.4.** With the notations above, the following are equivalent:

1. $R$ is a minimal intersection.
2. $R_p$ is a minimal intersection for all prime ideals $p$ of $Q$.
3. For all primes $p$ of $Q$,
\[
\text{depth}_{Q_p} Q_p + \text{depth}_{Q_p} R_p = \text{depth}_{Q_p} (R_1)_p + \text{depth}_{Q_p} (R_2)_p
\]
If $R_1$ and $R_1$ are Cohen-Macaulay, then (1)–(3) are equivalent to
4. $R_p$ is a proper intersection for all primes $p$ of $Q$. 
Recall that \(R\) is a called a proper intersection if \(\dim R = \dim R_1 + \dim R_2 - \dim Q\). Thus, in the Cohen-Macaulay case, part (4) of the theorem says that minimal intersections are proper intersections in a strong sense.

**Proof.** Suppose that \(R\) is a minimal intersection. We have \(R = Q/(I_1 + I_2)\) with \(Q\) a regular local ring, and \(I_1 I_2 = I_1 \cap I_2\). Let \(p\) be a prime ideal of \(Q\). Then \(R_p = Q_p/((I_1)_p + (I_2)_p)\) with \(Q_p\) a regular local ring. Thus \(R_p\) is a minimal intersection if and only if \((I_1)_p(I_2)_p = (I_1)_p \cap (I_2)_p\), but this follows easily from the fact that \(I_1 I_2 = I_1 \cap I_2\).

That (2) implies (3) is simply Lemma 3.2(2).

To show that (3) implies (1) we use 2.2 and 3.1:

\[
\sup\{ i \mid \text{Tor}_i^Q(R_1, R_2) \neq 0 \} = \sup\{ \text{depth}_{Q_p} Q_p - \text{depth}_{(R_1)_p} R_1 - \text{depth}_{(R_2)_p} R_2 \}
= \sup\{- \text{depth}_{Q_p} R_p \} = 0.
\]

The equivalence of (4) is clear, using Proposition 3.3. \(\square\)

It is useful to have a criteria for when a local ring is a minimal intersection. Recall that if \(X\) is a module over a local ring \(A\) with residue field \(\kappa\), then the Poincaré series of \(X\) over \(A\) is the formal power series \(P^A_X(t) = \sum_{i \geq 0} \dim \kappa \text{Tor}_i^A(X, \kappa)t^i\).

**Proposition 3.5.** The local ring \(R = Q/(I_1 + I_2)\) is a minimal intersection (with respect to \(Q\)) only if \(\frac{dP^Q_R}{dt}(-1) = 0\).

**Proof.** Since \(Q\) is a regular local ring, \(P^Q_{R_1}(t)\) and \(P^Q_{R_2}(t)\) are polynomials in \(t\), and since \(R_1\) and \(R_2\) are \(Q\)-modules of rank zero, we have \(P^Q_{R_1}(-1) = 0\) and \(P^Q_{R_2}(-1) = 0\). Now 3.1 shows that \(P^Q_R(t) = P^Q_{R_1}(t)P^Q_{R_2}(t)\). Thus \(P^Q_R(t)\) has \(-1\) as a double root. \(\square\)

**Remark.** The converse of Proposition 3.5 does not hold. Indeed, the Poincaré series over \(Q = k[[x, y]]\) of the local ring \(R = k[[x, y]]/(x^2, xy)\) has \(-1\) as a double root, yet \(R\) is not a minimal intersection with respect to \(Q\).

### 4. Vanishing over Minimal Intersections

This section contains the main results on non-trivial vanishing of Ext and Tor for modules over minimal intersections. The phenomenon of non-trivial vanishing is patterned on what happens over complete intersections, so we first briefly describe how non-trivial vanishing can occur in this case.

**Vanishing over Complete Intersections.** We first recall the following remarkable theorem of Avramov and Buchweitz [4], which makes use of support varieties, and which are reviewed below:

**4.1.** ([4]) Let \(M\) and \(N\) be finitely generated modules over a complete intersection \(R\). Then the following are equivalent.

1. \(\text{Tor}_i^R(M, N) = 0\) for all \(i \gg 0\);
2. \(\text{Ext}_i^R(M, N) = 0\) for all \(i \gg 0\);
3. \(\text{Ext}_i^R(N, M) = 0\) for all \(i \gg 0\);
4. \(V(M) \cap V(N) = \{0\}\).
Thus non-trivial vanishing occurs over complete intersections precisely when $M$ and $N$ are finitely generated modules, both of infinite projective dimension, such that $V(M) \cap V(N) = \{0\}$. We now describe a situation in which this trivial intersection of support varieties holds:

**Proposition 4.2.** Let $Q$ be a regular local ring, and $R = Q/(f_1, \ldots, f_r)$ a compete intersection of codimension $c \geq 2$. For $1 \leq r \leq c$, let $R_1 = Q/(f_1, \ldots, f_r)$ and $R_2 = Q/(f_{r+1}, \ldots, f_c)$. Suppose that $M'$ is a maximal Cohen-Macaulay module over $R_1$ and that $N'$ is a maximal Cohen-Macaulay module over $R_2$. For $M = M' \otimes_{R_1} R$ and $N = N' \otimes_{R_2} R$ we have

1. $V(M) \cap V(N) = \{0\}$
2. $\text{pd}_{R_1} M' = \text{pd}_R M$, and $\text{pd}_{R_2} N' = \text{pd}_R N$.

Thus non-trivial vanishing occurs whenever $M'$ and $N'$ are chosen to have infinite projective dimension over $R_1$ and $R_2$, respectively.

We briefly recall the definition of support variety (cf. [2]). Let $R$ be a complete intersection. We can without loss of generality assume that the residue field $k$ of $R$ is algebraically closed. For any finitely generated $R$-module $M$, the sequence of Ext modules $\text{Ext}_R^i(M, k)$ has the structure of finitely generated graded module over the polynomial ring $R = k[\chi_1, \ldots, \chi_c]$ of cohomology operators. Thus $\text{ann}_R \text{Ext}_R^i(M, k)$ is a homogeneous ideal of $R$, and we define the support variety of $M$, $V(M)$, to be the cone in $c$-dimensional affine space over $k$ defined by $\text{ann}_R \text{Ext}_R^i(M, k)$.

**Proof.** The proof is really that of [14, 3.1]; by construction, $M$ lifts to $M'$, and the proof of [14, 3.1] gives $(\chi_{r+1}, \ldots, \chi_c) \subseteq \text{ann}_R \text{Ext}_R^i(M, k)$. Thus we have $V((\chi_{r+1}, \ldots, \chi_c)) \supseteq V(M)$. Similarly, $V((\chi_1, \ldots, \chi_r)) \supseteq V(N)$, and so $V(M) \cap V(N) \subseteq V((\chi_1, \ldots, \chi_c)) \cap V((\chi_{r+1}, \ldots, \chi_c)) = \{0\}$.

There are two other relevant properties of vanishing Ext and Tor which hold over complete intersections. Both are well-known, and the first is referred to as the uniform Auslander condition in [17]. See [3, 4.2] and [13, 2.2] for the proofs.

**4.3.** Let $M$ and $N$ be finitely generated modules over a complete intersection $R$. Then

1. $\text{Ext}_R^i(M, N) = 0$ for all $i > 0$ if and only if $\text{Ext}_R^i(M, N) = 0$ for all $i > \min\{\text{depth } R - \text{depth } M, \text{depth } R - \text{depth } N\}$.
2. $\text{Tor}_R^i(M, N) = 0$ for all $i > 0$ if and only if $\text{Tor}_R^i(M, N) = 0$ for all $i > \min\{\text{depth } R - \text{depth } M, \text{depth } R - \text{depth } N\}$.

The second relevant property is a special case of what is proved in [4, 5.6]. Below we let $(\text{-})^*$ denote the dual $\text{Hom}_R(\text{-}, R)$.

**4.4.** Let $M$ and $N$ be finitely generated modules over a complete intersection $R$. Then

1. $\text{Ext}_R^i(M, N) = 0$ for all $i > 0$ if and only if $\text{Ext}_R^i(M, N^*) = 0$ for all $i > 0$.
2. $\text{Tor}_R^i(M, N) = 0$ for all $i > 0$ if and only if $\text{Tor}_R^i(M, N^*) = 0$ for all $i > 0$.

We can generalize these aspects of non-trivial vanishing to minimal intersections. The trade-off to considering a class of rings much more general than the complete intersections is that we establish non-trivial vanishing for specific classes of modules...
Vanishing over Arbitrary Minimal Intersections. We assume now that $R = Q/(I_1 + I_2)$ is a minimal intersection with $Q$ a regular local ring, and $R_1 = Q/I_1$ and $R_2 = Q/I_2$. The following theorem is the main result of the paper.

**Theorem 4.5.** Let $R$ be a minimal intersection, $M'$ any sufficiently high syzygy module over $R_1$ of a finitely generated $R_1$-module, and $N'$ any sufficiently high syzygy module over $R_2$ of a finitely generated $R_2$-module. Let $L'$ any sufficiently high cosyzygy module over $R_2$ of a finitely generated $R_2$-module. Then for $M = M' \otimes_{R_1} R$, $N = N' \otimes_{R_2} R$, and $L = \text{Hom}_Q(R_1, L')$ the following hold:

1. $\text{Tor}_i^R(M, N) = 0$ for all $i > \dim Q$;
2. $\text{Ext}_i^R(M, L) = 0$ for all $i > \dim Q$;
3. $\text{pd}_R M = \text{pd}_{R_1} M'$, $\text{pd}_R N = \text{pd}_{R_2} N'$, id$_R L = \text{id}_{R_2} L'$.

**Proof.** Let $N''$ be any finitely generated $R_2$-module. Since $Q$ is a regular local ring we have $\text{Tor}_i^Q(R_1, N'') = 0$ for all $i \geq 0$. Take an exact sequence $0 \to N' \to R_2^i \to N'' \to 0$. Then from the derived long exact sequence of Tor

$$\cdots \to \text{Tor}_i^Q(R_1, N') \to \text{Tor}_i^Q(R_1, R_2) \to \text{Tor}_i^Q(R_1, N'') \to \cdots,$$

and the fact that $\text{Tor}_i^Q(R_1, R_2) = 0$ for all $i \geq 1$, we have the isomorphisms $\text{Tor}_{i+1}^Q(R_1, N'') \cong \text{Tor}_i^Q(R_1, N')$ for all $i \geq 1$. Thus if $N'$ is a sufficiently high syzygy over $R_2$, we may assume that

$$\text{Tor}_i^Q(R_1, N') = 0$$

for all $i \geq 1$. Applying 2.1(1) we get the isomorphisms

$$\text{Tor}_i^Q(M', N') \cong \text{Tor}_i^R(M', N' \otimes_Q R_1) = \text{Tor}_{i+1}^R(M', N)$$

for all $i$. Note that $\text{Tor}_i^Q(R_1, R_2) = 0$ for all $i \geq 1$ implies that a minimal free resolution of $R$ over $R_1$ is attained by tensoring a minimal free resolution of $R_2$ over $Q$ with $R_1$. Thus $R$ has finite projective dimension over $R_1$. By choosing a sufficiently high syzygy $M'$ over $R_1$ we can assume that

$$\text{Tor}_i^{R_1}(M', R) = 0$$

for all $i \geq 1$. By 2.1(1) we have the isomorphisms

$$\text{Tor}_i^{R_1}(M', N) \cong \text{Tor}_i^R(M' \otimes_{R_1} R, N) = \text{Tor}_i^R(M, N)$$

for all $i$. Thus $\text{Tor}_i^R(M, N) = 0$ for all $i > \dim Q$, and this establishes the claim about the vanishing of homology.

To see the statements regarding the projective dimensions of $M$ and $N$, note that by (4.5.2) a minimal free resolution of $M$ over $R$ is obtained by tensoring a minimal free resolution of $M'$ over $R_1$ with $R$. Thus $\text{pd}_R M = \text{pd}_{R_1} M'$. By symmetry we have $\text{pd}_R N = \text{pd}_{R_2} N'$.

For (2), let $L''$ be an $R_2$-module. Since $Q$ is a regular local ring, $\text{Ext}_i^Q(R_1, L'') = 0$ for all $i \geq 0$. Let $0 \to L'' \to I \to L' \to 0$ be an exact sequence of $R_2$-modules with $I$ injective. Then $I$ is a direct sum of injective hulls $E_{R_2}(R_2/p)$ of quotients $R_2/p$ with $p$ a prime ideal of $R_2$. If $P$ is a prime ideal of $Q$ which is a preimage of $p$, then $E_{R_2}(R_2/p) = \text{Hom}_Q(R_2, E_Q(Q/P))$, where $E_Q(Q/P)$ is the injective hull
We have the isomorphisms $\text{Ext}^i_{Q/P}(R_1, I)$ for all $i \geq 1$, and so from the long exact sequence of Ext
\[
\cdots \to \text{Ext}^i_Q(R_1, L'') \to \text{Ext}^i_Q(R_1, I) \to \text{Ext}^i_Q(R_1, L') \to \cdots
\]
we get
\[
\text{Ext}^i_Q(R_1, L') \cong \text{Ext}^{i+1}_Q(R_1, L'')
\]
for all $i \geq 1$. Now it is clear that we can replace $L''$ by an $R_2$-module $L'$ such that
\[
(4.5.3) \quad \text{Ext}^i_Q(R_1, L') = 0 \quad \text{for all} \quad i \geq 1.
\]
By 2.1(2) we have the isomorphisms
\[
\text{Ext}^i_Q(M', L') \cong \text{Ext}^i_{R_1}(M', \text{Hom}_Q(R_1, L')) = \text{Ext}^i_{R_1}(M', L)
\]
for all $i$. As in the part of the proof for (1) above, we can choose a finitely generated $R_1$-module $M'$ such that $\text{Tor}^1_{R_1}(M', R) = 0$ for all $i \geq 1$, which by 2.1(1) gives
\[
\text{Ext}^i_{R_1}(M', L) \cong \text{Ext}^i_R(M' \otimes_{R_1} R, L) = \text{Ext}^i_R(M, L)
\]
for all $i$. Therefore we have $\text{Ext}^i_R(M, L) = 0$ for all $i > \dim Q$.

To finish the proof we just need to justify that $\text{id}_R L = \text{id}_{R_2} L'$. By 3.1, $\text{Tor}^1_Q(R_1, R_2) = 0$ for all $i \geq 1$. Then 2.1(1) shows that $\text{Ext}^i_Q(R_1, L') \cong \text{Ext}^i_{R_2}(R, L')$ for all $i$, in particular, $L \cong \text{Hom}_{R_2}(R, L')$. Then by (4.5.3) we have $\text{Ext}^i_{R_2}(R, L') = 0$ for all $i \geq 1$. Thus a minimal injective resolution of $L$ over $R$ is obtained by applying $\text{Hom}_{R_2}(R, -)$ to a minimal injective resolution of $L'$ over $R_2$, and so $\text{id}_R L = \text{id}_{R_2} L'$.

\[
\square
\]

Over Cohen-Macaulay minimal intersections we can establish non-trivial vanishing of Tor for a larger class of modules, and non-trivial vanishing of Ext for pairs of finitely generated modules. Indeed, over a Cohen-Macaulay ring a higher syzygy module is maximal Cohen-Macaulay, but a maximal Cohen-Macaulay module need not be a higher syzygy module.

Note that for the classes of modules identified in the following corollary, Property 4.3 holds.

Recall from 3.3 that $R$ is Cohen-Macaulay if and only if both $R_1$ and $R_2$ are Cohen-Macaulay. We let $(-)\dual$ denote the dual $\text{Hom}_R(-, \omega)$, were $\omega$ is the canonical module of $R$.

**Corollary 4.6.** Let $R$ be a Cohen-Macaulay minimal intersection. Suppose that $M'$ is a maximal Cohen-Macaulay $R_1$-module, and $N'$ is a maximal Cohen-Macaulay $R_2$-module. Then for $M = M' \otimes_{R_1} R$ and $N = N' \otimes_{R_2} R$ we have

1. $M$, $N$, and $M \otimes_R N$ are maximal Cohen-Macaulay $R$-modules;
2. $\text{Tor}^i_R(M, N) = 0$ for all $i \geq 1$;
3. $\text{Ext}^i_R(M, N') = 0$ for all $i \geq 1$;
4. $\text{pd}_R M = \infty$ if and only if $M'$ is not free; $\text{pd}_R N = \infty$ if and only if $N'$ is not free if and only if $\text{id}_R N' = \infty$.

**Proof.** (1). From Theorem 3.4 we have that
\[
\text{depth}_{Q_p} Q_p - \text{depth}_{Q_p} (R_1)_p - \text{depth}_{Q_p} (R_2)_p = - \text{depth}_{Q_p} R_p \leq 0
\]
for all primes $p$ of $Q$. Since $M'$ is a maximal Cohen-Macaulay $R_1$-module and $N'$ is a maximal Cohen-Macaulay $R_2$-module, $\text{depth}_{Q_p}(R_1)_p = \text{depth}_{Q_p} M'_p$ and $\text{depth}_{Q_p}(R_2)_p = \text{depth}_{Q_p} N'_p$ for all primes $p$ of $Q$. Thus for all primes $p$ of $Q$,

$$\text{depth}_{Q_p} Q_p - \text{depth}_{Q_p} M'_p - \text{depth}_{Q_p} N'_p = \text{depth}_{Q_p} Q_p - \text{depth}_{Q_p}(R_1)_p - \text{depth}_{Q_p}(R_2)_p \leq 0$$

Thus by 2.2 we obtain

$$(4.6.1) \quad \text{Tor}_{i}^{Q}(M', N') = 0 \quad \text{for all} \quad i \geq 1.$$  

It follows that $\text{pd}_{Q}(M' \otimes_{Q} N') = \text{pd}_{Q} M' + \text{pd}_{Q} N'$. Now the Auslander-Buchsbaum formula gives the equation

$$\text{depth}_{Q} Q + \text{depth}_{Q}(M' \otimes_{Q} N') = \text{depth}_{Q} M' + \text{depth}_{Q} N'$$

Using the fact that $M'$ and $N'$ are maximal Cohen-Macaulay, and comparing with $\text{depth}_{Q} Q + \text{depth}_{Q} R = \text{depth}_{Q} R_1 + \text{depth}_{Q} R_2$ from 3.2, we see that $\text{depth}_{Q} R = \text{depth}_{Q}(M \otimes_{Q} N)$, and this is the same as $\text{depth}_{R} R = \text{depth}_{R}(M \otimes_{R} N)$.

The same proof shows that $M$ and $N$ are both maximal Cohen-Macaulay, just by replacing $N'$ by $R_2$, and $M'$ by $R_1$, respectively.

(2). Following the proof of Theorem 4.5(1), and in light of (4.6.1), we just need to show that (4.5.1) and (4.5.2) hold. As in the argument for part (1), we have $\text{Tor}_{i}^{Q}(R_1, N') = 0$ for all $i \geq 1$, which is (4.5.1). Similarly, $\text{Tor}_{i}^{Q}(M', R_2) = 0$ for all $i \geq 1$, and since $\text{Tor}_{i}^{Q}(R_1, R_2) = 0$ for all $i \geq 1$, 2.1(1) implies we also have $\text{Tor}_{i}^{R_1}(M', R) = 0$ for all $i \geq 1$, which is (4.5.2).

Property (3) follows from (1), (2), and Proposition 2.4.

By Theorem 4.5(3,4) the only part of (4) we need to show is the last statement. We have $N'$ is free over $R_2$ if and only if $\text{Tor}_{i}^{R}(k, N) = 0$ for all $i \gg 0$ if and only if (by Proposition 2.4) $\text{Ext}_{R}^{i}(k, N^\vee) = 0$ for all $i \gg 0$ if and only if $\text{id}_{R} N^\vee < \infty$. □

Remark. The plentitude of modules involved in non-trivial vanishing according to Corollary 4.6 thus depends on the number of non-isomorphic indecomposable maximal Cohen-Macaulay modules over $R_1$ and $R_2$. Much work has been done on the classification of Cohen-Macaulay rings having only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules, the so-called rings of finite Cohen-Macaulay type. See [21] for a survey of the subject. In particular, a Cohen-Macaulay ring of finite Cohen-Macaulay type has at most an isolated singularity [10]. Outside of this case, the literature suggests that the number of non-isomorphic indecomposable maximal Cohen-Macaulay modules over a Cohen-Macaulay ring is quite large (see, for example, [6]).

A stronger analogy to vanishing over complete intersections is attained when we assume that $R$ is Gorenstein: Properties (4) and (5) below mimic 4.4, and (6) that of 4.1(3).

Recall by Proposition 3.3 that $R$ is Gorenstein if and only if both $R_1$ and $R_2$ are Gorenstein. Below we let $(-)^*$ denote the dual $\text{Hom}_{R}(-, R)$.

**Corollary 4.7.** Let $R$ be a Gorenstein minimal intersection. Suppose that $M'$ is a maximal Cohen-Macaulay $R_1$-module, and $N'$ is a maximal Cohen-Macaulay $R_2$-module. Then for $M = M' \otimes_{R_1} R$ and $N = N' \otimes_{R_2} R$ we have

1. $M$, $N$, and $M \otimes_{R} N$ are maximal Cohen-Macaulay $R$-modules;
2. $\text{Tor}_{i}^{R}(M, N) = 0$ for all $i \geq 1$;
(3) $\text{Ext}^i_R(M, N^*) = 0$ for all $i \geq 1$;
(4) $\text{Tor}_i^R(M, N^*) = 0$ for all $i \geq 1$;
(5) $\text{Ext}^i_R(M, N) = 0$ for all $i \geq 1$;
(6) $\text{Ext}^i_R(N, M) = 0$ for all $i \geq 1$;
(7) $\text{pd}_R M = \infty$ if and only if $M'$ is not free, and $\text{pd}_R N^* = \infty$ if and only if $N'$ is not free.

Proof. Properties (1)-(3) and (7) are handled by Corollary 4.6. For (4), (5), and (6) it suffices to show that $N^* \cong \text{Hom}_{R_2}(N', R_2) \otimes_{R_2} R$. But this is exactly the statement of Proposition 2.3.

5. Examples and a Sufficient Condition

The following is an example illustrating that non-trivial vanishing can occur over rings which are not minimal intersections.

Example 5.1. Let $Q = k[[W, X, Y, Z]]$ where $k$ is a field, and

$$R = Q/(W^2, X^2, Z^2, XY, WX + XZ, WY + YZ, Y^2 - WZ).$$

Then one may check that $R$ is a zero-dimensional local ring with $P^Q_R(t) = 1 + 7t + 13t^2 + 10t^3 + 3t^4$. According to Proposition 3.5, $R$ is not a minimal intersection with respect to $Q$. Let $w$ denote the image of $W$ in $R$, etc. Consider the finitely generated $R$-modules $M = \text{coker} \phi$, where $\phi$ is represented with respect to the standard basis of $R^2$ by the matrix

$$
\begin{pmatrix}
-w & x \\
-y & z
\end{pmatrix},
$$

and $N = R$. Then we have $\text{Ext}_R^i(M, N) = 0$ for all $i > 0$. Moreover, $\text{pd}_R M = \infty$, and $\text{id}_R N = \infty$.

The following example shows that Proposition 4.2 does not describe the only way non-trivial vanishing occurs over complete intersections. The details of the example are proven in [15].

Example 5.2. Let $Q = k[[V, W, X, Y, Z]]$, with $k$ a field, and $R = Q/(VW, XY)$. Then $R$ is a codimension two complete intersection. Let $v$ denote the image of $V$ in $R$, etc., and $M$ be the cokernel of the map $\varphi : R^8 \to R^8$ represented with respect to the standard basis of $R^8$ by the matrix

$$
\begin{pmatrix}
-v & 0 & 0 & -z & 0 & 0 & 0 & y \\
-w & 0 & -z & 0 & 0 & 0 & y & 0 \\
0 & 0 & v & -w & 0 & y & 0 & 0 \\
0 & 0 & 0 & w & y & 0 & 0 & 0 \\
0 & -w & 0 & x & 0 & 0 & 0 & 0 \\
0 & -w & x & 0 & 0 & 0 & 0 & 0 \\
0 & y & 0 & 0 & 0 & 0 & 0 & 0 \\
x & z & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

For $N = R/(v)$, we have $\text{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. Moreover, $M$ is not of the form described in Proposition 4.2. That is, for no minimal generator $f$ of $(VW, XY)$ is there a maximal Cohen-Macaulay $R_1 = Q/(f)$-module $M'$ such that $M \cong M' \otimes_{R_1} R$.
A Sufficient Condition. Let $R = Q/(I_1 + I_2)$ be a minimal intersection with $Q$ a regular local ring, and $R_1 = Q/I_1$, $R_2 = Q/I_2$. In this section we discuss a sufficient condition for determining whether a finitely generated $R$-module $M$ has a syzygy over $R$ of the form $M' \otimes_{R_1} R$ for some $R_1$-module $M'$ of infinite projective dimension over $R_1$ satisfying $\text{Tor}^{R_1}_i(M', R) = 0$ for all $i \geq 1$, and hence of the form identified in Theorem 4.5.

Let

$$F : \cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to M \to 0$$

be a minimal $R$-free resolution of $M$. Choose a sequence of free $Q$-modules $\tilde{F}_i$ and maps $\tilde{\partial}_i$ between them

$$\tilde{F} : \cdots \to \tilde{F}_2 \xrightarrow{\tilde{\partial}_2} \tilde{F}_1 \xrightarrow{\tilde{\partial}_1} \tilde{F}_0 \to 0$$

such that $F$ and $\tilde{F} \otimes_Q R$ are isomorphic complexes. It is useful to think of the maps $\partial_i$ as being given by matrices over $R$ (with respect to some fixed bases of the $F_i$), in which case the maps $\tilde{\partial}_i$ may be thought of as matrices of preimages in $Q$ of the entries of the matrices representing the $\partial_i$. Since $F$ is a complex of $R$-modules we have $\tilde{\partial}_{i-1} \tilde{\partial}_i \equiv 0$ modulo $I_1 + I_2$, in other words $(\tilde{\partial}_{i-1} \otimes_Q R)(\tilde{\partial}_i \otimes_Q R) = 0$. For the sufficient condition given below we will be considering the sequences of maps

$$(5.2.1) \quad \tilde{F}_i \otimes_Q R_j \xrightarrow{\tilde{\partial}_i \otimes R_j} \tilde{F}_{i-1} \otimes_Q R_j \xrightarrow{\tilde{\partial}_{i-1} \otimes R_j} \tilde{F}_{i-2} \otimes_Q R_j$$

for $j = 1, 2$ and $i \geq 2$.

**Proposition 5.3.** Let $M$ be a finitely generated $R$-module of infinite projective dimension over $R$, and suppose $(\tilde{F}, \tilde{\partial})$ is some lifting to $Q$ of a minimal $R$-free resolution $(F, \partial)$ of $M$. If the sequence of maps (5.2.1) forms an exact sequence for some $i \geq 2$, then $M$ has a syzygy over $R$ of the form $M' \otimes R_j$ where $M'$ is an $R_j$ module satisfying $\text{Tor}^{R_j}_l(M', R) = 0$ for all $l \geq 1$, and hence $M$ participates in a non-trivial vanishing of all higher Tor.

**Proof.** Without loss of generality assume that $j = 1$, and that (5.2.1) forms an exact sequence for fixed $i \geq 2$. Let $M'_{i-2} := \text{coker}(\tilde{\partial}_{i-1} \otimes R_1)$. Then

$$\tilde{F}_i \otimes Q R_1 \xrightarrow{\tilde{\partial}_i \otimes R_1} \tilde{F}_{i-1} \otimes Q R_1 \xrightarrow{\tilde{\partial}_{i-1} \otimes R_1} \tilde{F}_{i-2} \otimes Q R_1 \to M'_{i-2} \to 0$$

is the beginning of an $R_1$-free resolution of $M'_{i-2}$. Tensoring this complex with $R$ we get $F_i \xrightarrow{\partial_i} F_{i-1} \xrightarrow{\partial_{i-1}} F_{i-2}$, which is exact. This means that $\text{Tor}^{R_1}_i(M'_{i-2}, R) = 0$. Since $\text{Tor}^{Q}_l(R_1, R_2) = 0$ for all $i \geq 1$, by 2.1(1) we have $\text{Tor}^{R_1}_l(M'_{i-2}, R) \cong \text{Tor}^{Q}_l(M'_{i-2}, R_2)$ for all $l \geq 1$. Therefore, by rigidity of Tor for regular local rings, $\text{Tor}_l^{R_1}(M'_{i-2}, R) = 0$ for all $l \geq 1$. This finishes the proof, since $M'_{i-2} \otimes R_1 R \cong \text{coker} \partial_{i-1}$ is a syzygy of $M$ over $R$. \hfill $\Box$

**Remark.** Suppose $j = 1$. If $I_2$ happens to be generated by a $Q$-regular sequence, then we a priori only need to know that the sequence of maps in (5.2.1) with $j = 1$ forms a complex in order to invoke the conclusion of Proposition 5.3. For if (5.2.1) forms a complex with $j = 1$, and $I_2$ is generated by a $Q$-regular sequence, then this sequence is also regular on $R_1$, and by working our way inductively from $R$ up to $R_1$, Nakayama’s lemma yields that (5.2.1) is exact.
Next we give examples using Macaulay 2 which illustrate Proposition 5.3. We first discuss a few details of the liftings ($\tilde{\mathbf{F}}, \tilde{\partial}$), and define special maps based on the notion of Eisenbud operators, which were developed in [7] for finitely generated modules over a complete intersection.

Fix a minimal generating set $f_1, \ldots, f_c$ of $I_1 + I_2$ such that $I_1$ is generated by $f_1, \ldots, f_r$ and $I_2$ generated by $f_{r+1}, \ldots, f_c$. (By our assumption that $I_1$ and $I_2$ are non-zero, we have $1 \leq r \leq c-1$.) Since the products $\tilde{\partial}_{i-1} \tilde{\partial}_i$ are zero modulo $I_1 + I_2$, we may express them in terms of the $f_j$: write

$$\tilde{\partial}_{i-1} \tilde{\partial}_i = \sum_{j=1}^c f_j \tilde{t}_{i,j}, \quad (5.3.1)$$

where the $\tilde{t}_{i,j}$ are maps $\tilde{t}_{i,j} : \tilde{F}_i \rightarrow \tilde{F}_{i-2}$. Note that these maps are not uniquely defined. They depend first on the resolution $\mathbf{F}$, then on the lifting $(\mathbf{F}, \partial)$, and then on the choice of the expression in (5.3.1).

In investigating when the sequence (5.2.1)

$$F_i \otimes_Q R_j \xrightarrow{\partial_i \otimes R_j} \tilde{F}_{i-1} \otimes_Q R_j \xrightarrow{\partial_{i-1} \otimes R_j} \tilde{F}_{i-2} \otimes_Q R_j$$

is exact, we proceed in two steps. First we need to know when it forms a complex. For $j = 1$ this is implied by the condition

$$\tilde{t}_{i,r+1} \otimes R_1 = \cdots = \tilde{t}_{i,c} \otimes R_1 = 0, \quad (5.3.2)$$

and for $j = 2$ the condition

$$\tilde{t}_{i,1} \otimes R_2 = \cdots = \tilde{t}_{i,r} \otimes R_2 = 0. \quad (5.3.3)$$

Once we know conditions (5.3.2) or (5.3.3) hold, we compute the homology of the corresponding complex (5.2.1) to see that it is zero.

In the following examples, we perform both steps using Macaulay 2. For the first step we use a special script, which can be obtained from the authors, called getEisoplist which computes the maps $\tilde{t}_{i,j}$ and stores them as a list of lists called Eisoplist. Because the internal indexing used by Macaulay 2 starts at 0, the element Eisoplist[i][j] actually represents the map $\tilde{t}_{i+2,j+1}$. The code may also compute the Eisenbud operators, which are the maps $t_{i,j} = t_{i,j} \otimes_Q R$ defined in [7] in the case where $R$ is a complete intersection.

The input for this script is a chain complex and an integer. Presumably, the chain complex is a free resolution $(\mathbf{F}, \partial)$ over $R$ of the module $M$, and this may be obtained simply by using the res command in Macaulay 2. The integer tells the script up to what degree $i$ the maps $\tilde{t}_{i,j}$ should be computed. The lifting $(\mathbf{F}, \partial)$ of the given resolution $(\mathbf{F}, \partial)$ is done in the script using the Macaulay 2 command lift. Finally, the choice of the $\tilde{t}_{i,j}$ defined by expression (5.3.1) is decided in the script using the //Igb command in Macaulay 2, where Igb is a Gröbner basis of the ideal $(f_1, \ldots, f_c)$.

**Example 5.4.** Let $Q = \mathbb{Q}[x, y, z]$ and $R = Q/I$, where

$$I := (x^2 - yz, xz - y^2, z^2 - xy, x^2 + yz).$$

Then $R$ is a zero-dimensional minimal intersection, and the $R$-module $M = R/(x + y + z)$ participates in non-trivial vanishing of all higher Tor.
We first load the script `getEisoplist`, then show that $R$ is in fact an minimal
intersection by testing $\text{Tor}_1^Q(Q/I_1, Q/I_2) = 0$, then exhibit a minimal resolution of
$M$, showing that it has infinite projective dimension over $R$.

```plaintext
i1 : load"getEisoplist.m2"
--loaded getEisoplist.m2
i2 : Q = QQ[x,y,z];
i3 : I = ideal(x^2-y*z,x*z-y^2,z^2-x*y,x^2+y*z);
o3 : Ideal of Q
i4 : Tor_1(coker matrix{{x^2-y*z,x*z-y^2,z^2-x*y,z^2+y*z}},coker matrix{{x^2+y*z}}) == 0
o4 = true
i5 : R = Q/I
o5 = R
o5 : QuotientRing
i6 : M = coker matrix{{x+y+z}}
o6 = cokernel | x+y+z |
o6 : R-module, quotient of R
i7 : Mres = res(M,LengthLimit=>6)
1 1 2 4 8 16 32
o7 = R <-- R <-- R <-- R <-- R <-- R <-- R
0 1 2 3 4 5 6
o7 : ChainComplex
```

Now we compute the maps $\tilde{t}_{i,j}$. What is shown is \{\tilde{t}_{2,1}, \tilde{t}_{2,2}, \tilde{t}_{2,3}, \tilde{t}_{2,4}\}. Notice
that $\tilde{t}_{2,4} = 0$, and so it is also zero modulo $I_1$.

```plaintext
i8 : MEisoplist = getEisoplist(Mres,2)
o8 = {{{2} | 0 1 |, {2} | -1 0 |, {2} | -1 -1 |, 0}}
o8 : List
```

The next step is check that the homology of the complex

$$F_2 \otimes_Q R_1 \xrightarrow{\partial_2 \otimes R_1} F_1 \otimes_Q R_1 \xrightarrow{\partial_1 \otimes R_1} F_0 \otimes_Q R_1$$

is zero (although we do not really need this step, by the remark following Proposition 5.3, since $I_2 = (x^2 + yz)$ is generated by a regular element). First we need to define the ring $R_1$.

```plaintext
i9 : use Q
o9 = Q
o9 : PolynomialRing
i10 : R1 = Q/ideal(x^2-y*z,x*z-y^2,z^2-x*y)
o10 = R1
```
Therefore, by Proposition 5.3, \( M \) participates in non-trivial vanishing.

We can build a companion module \( N \) for \( M \) as per Theorem 4.5, which yield non-trivial vanishing of all higher \( \text{Tor}_i^R(M, N) \). The steps below are: define \( R_2 \), resolve the residue field over this ring, take an appropriate syzygy, and tensor this syzygy down to the ring \( R \).

```plaintext
i12 : use Q;
i13 : R2 = Q/ideal(x^2+y*z)
o13 = R2
  o13 : QuotientRing

i14 : Ntres = res coker vars R2
  1 3 4 4 4
o14 = R2 <-- R2 <-- R2 <-- R2 <-- R2
  0 1 2 3 4
  o14 : ChainComplex

i15 : N = (coker lift(Ntres.dd_3,Q)) ** R
  o15 = cokernel {2} | x z 0 x |
       {2} | -y x 0 -y |
       {2} | z 0 x 0 |
       {2} | 0 z y x |
      4
  o15 : R-module, quotient of R

The beginning of a minimal resolution of \( N \) over \( R \) is given to show that \( \text{pd}_R N = \infty \). Afterwards we compute \( \{\tilde{t}_{2,1}, \tilde{t}_{2,2}, \tilde{t}_{2,3}, \tilde{t}_{2,4}\} \) for \( N \). Note that \( \tilde{t}_{2,1} = \tilde{t}_{2,2} = \tilde{t}_{2,3} = 0 \)

```plaintext

```plaintext
i16 : Nres = res(N,LengthLimit=>6)
  4 4 4 4 4 4 4
o16 = R <-- R <-- R <-- R <-- R <-- R <-- R
  0 1 2 3 4 5 6
  o16 : ChainComplex

i17 : NEisoplist = getEisoplist(Nres,2)
  o17 = {{0, 0, 0, (2) | 1 0 0 1 |},
         (2) | 0 0 0 1 |}
  o17 : List

Finally, we compute the homology of the corresponding complex to show that it is zero. We also show that indeed the first several \( \text{Tor}_i^R(M, N) \) are zero.
```
Example 5.5. Let \( Q = \mathbb{Q}[u,v,w,x,y,z] \) and \( R = Q/I \), where
\[
I = (uv-vx,uw-uz-wx+xz,vw-vz,u^2-v^2-2ux+x^2,\]
\[
v^2-w^2+2wz-z^2,xy-vx,xz,yz-vz,\]
\[
x^2-y^2+2vy-v^2+v^2+2vy-z^2).
\] Then \( R \) is a zero-dimensional Gorenstein minimal intersection.

If we let \( I_1 \) be generated by the first five generators of \( I \) and \( I_2 \) generated by the second five, then we exhibit that \( R \) is an minimal intersection.

The last map in the following resolution of \( I \) over \( Q \) shows that in fact \( R \) is Gorenstein.
MINIMAL INTERSECTIONS AND VANISHING (CO)HOMOLOGY

i26 : M = coker matrix({u-x,v,w-z});
i27 : Mres = res(M,LengthLimit=>6)
        1 3 8 21 55 144 377
      o27 = R <-- R <-- R <-- R <-- R <-- R <-- R
           0 1 2 3 4 5 6
      o26 : ChainComplex

i28 : tM = getEisoplist(Mres,2);

i29 : tM#0#5,tM#0#6,tM#0#7,tM#0#8,tM#0#9
      o29 = (0, 0, 0, 0, 0)
      o29 : Sequence

i30 : use Q;
i31 : R1 = Q/ideal(u*v-v*x,u*w-u*z-w*x+x*z,v*w-v*z,u^2-v^2-2*u*x+x^2,v^2-w^2+2*w*z-z^2);
i32 : homology(lift(Mres.dd_1,Q) ** R1,lift(Mres.dd_2,Q) ** R1) == 0
      o32 = true

Now we identify a companion module \( N \) for \( M \) such that the pair has non-trivial vanishing of all higher \( \text{Tor} \). We compute the \( \tilde{t}_{i,j} \) for \( N \) and show that \( \tilde{t}_{2,1} = \tilde{t}_{2,2} = \tilde{t}_{2,3} = \tilde{t}_{2,4} = \tilde{t}_{2,5} = 0 \). Then we show that the corresponding complex has zero homology.

i33 : use R;
i34 : N = coker matrix({x,y-v,z});
i35 : Nres = res(N,LengthLimit=>6)
        1 3 8 21 55 144 377
      o35 = R <-- R <-- R <-- R <-- R <-- R <-- R
           0 1 2 3 4 5 6
      o35 : ChainComplex
i36 : tN = getEisoplist(Nres,2);

i37 : tN#0#0,tN#0#1,tN#0#2,tN#0#3,tN#0#4
      o37 = (0, 0, 0, 0, 0)
      o37 : Sequence
i38 : use Q;
i39 : R2 = Q/ideal(x*y-v*x,z*y-z-v*z,z^2-y^2+2*y+v^2+v^2*y^2-v^2-2*v*y-z^2);
i40 : homology(lift(Nres.dd_1,Q) ** R2,lift(Nres.dd_2,Q) ** R2) == 0
      o40 = true

Finally, we compute the first few \( \text{Tors} \), and then following Theorem 4.7, we show that the first few \( \text{Ext}^i_R(M, \text{Hom}_R(N, R)) \) vanish.

i41 : Tor_1(M,N) == 0,Tor_2(M,N) == 0,Tor_3(M,N) == 0
\[ o_{41} = (true, true, true) \]
\[ o_{41} : \text{Sequence} \]
\[ i_{42} : \text{Hom}(N, R) \]
\[ o_{42} = \text{image} \mid z_2 \]
\[ o_{42} : R\text{-module, submodule of } R \]
\[ i_{43} : \text{Ext}^1(M, \text{Hom}(N, R)) = 0, \text{Ext}^2(M, \text{Hom}(N, R)) = 0, \text{Ext}^3(M, \text{Hom}(N, R)) = 0 \]
\[ o_{43} = (true, true, true) \]
\[ o_{43} : \text{Sequence} \]

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