A family of self-avoiding walks on the Sierpiński gasket

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1. Pre-Sierpiński gasket

Let $F_0$ be a triangle $\triangle Oab$, $F_N$ be the graph with edges of length $2^{-N}$, and $F = \text{cl}(\bigcup_{N=0}^{\infty} F_N)$ be a (finite) Sierpiński gasket.

- Next, define self-avoiding paths on $F_N$ and probability measures on the path spaces inductively.
Self-avoiding paths

\( w \): self-avoiding path on \( F_N \) if

\[
\begin{align*}
  w(0) &= O, \quad (w(i), w(i + 1)) \in \{ \text{edges on } F_N \}, \\
  w(i) &\in \{ \text{vertices on } F_N \}, \quad w(i) \neq w(j) \ (i \neq j), \quad \text{and } w(\ell(w)) = a.
\end{align*}
\]

\[
a = \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right)
\]

\( O = (0, 0) \quad b = (1, 0) \)
2. A set of probability measures with multi parameter

Self-avoiding paths on $F_1$:

Given $p_i, q_i \geq 0$, $\sum_{i=1}^{10} p_i = 1$, $\sum_{i=1}^{10} q_i = 1$, $p_8, p_9, p_{10} = 0$, define

$$P^1_1[w_i] = p_i, \quad P^2_1[w_i] = q_i.$$
Branching

Idea: $P_{N+1}^1$ and $P_{N+1}^2$ are obtained by the following branching.

Following this idea, define $X_N(w)(j) = w(j)$, $j = 0, 1, \ldots, \ell(w)$, where $w$ is a self-avoiding path on $F_N$.

(Suppose that the first branching follows $P_1^1$. )
Example

Loop-erased random walk (defined by erasing loops in descending order of the size of loops from a simple random walk) on $F_N$ (Hattori, Mizuno 14’) is included as a special case.

$$p_1 = \frac{1}{2}, \ p_2 = p_3 = p_7 = \frac{2}{15}, \ p_4 = p_5 = p_6 = \frac{1}{30}, \ p_8 = p_9 = p_{10} = 0,$$

$$q_1 = \frac{1}{9}, \ q_2 = q_3 = q_7 = \frac{11}{90}, \ q_4 = q_5 = q_6 = \frac{2}{45}, q_7 = \frac{8}{45}, q_8 = \frac{2}{9},$$

$$q_9 = q_{10} = \frac{1}{18}.$$

Additionally, ‘standard’ self-avoiding walk (HHK 91’), loop-erased self-repelling walk (HOO, 17’) are included as special cases respectively (we omit details here).

Assigning different values to $p_i, q_i$ gives different SAWs.
3. Scaling limit

Let \( p = (p_1, \ldots, p_{10}, q_1, \ldots, q_{10}) \) and \( X_N(w)(t) = w(t), \ t \in [0, \infty) \): a self avoiding walk on \( F_N \) (linear interpolated).

**Theorem (scaling limit)**

*For any \( p \), there exists \( \lambda = \lambda(p) (2 \leq \lambda \leq 3) \) such that*

\[
X_N(\lambda^N t) \rightarrow X(t) \quad \text{a.s. as } \ N \rightarrow \infty.
\]

*and \( d_H = \log \lambda / \log 2 \) (Hausdorff dimension of the path) with probability 1.*

- \( d_H \) takes any values from 1 to \( \log 3 / \log 2 \).
- \( X_N \) is self-avoiding, but \( X = X(p) \) is not necessarily self-avoiding and the number of triangles produced at each branching affects the self-avoiding property.
Two extream cases

- If $p_1 + p_2 + p_3 + p_4 = q_1 + q_2 + q_3 + q_4 = 1$, then, with probability 1, $d_H = 1$ and $X$ is an uniform linear motion.

- If $p_5 + p_6 + p_7 = q_5 + \cdots + q_{10} = 1$, then, with probability 1, $d_H = \log 3/\log 2$, that is, $X$ fills the state space (the SG), and the speed of the motion is constant. We call it Peano curve.
4. Self-avoiding property

**Theorem (Self-avoiding)**

If $p_5 + p_6 + p_7 < 1$, $q_5 + \cdots + q_{10} < 1$, then the scaling limit $X$ is self-avoiding.
Asymptotic behavior

Suppose that $p_5 + p_6 + p_7 < 1, q_5 + \cdots + q_{10} < 1$. In this case, we have some results about asymptotic behavior.

**Theorem (Short time behavior)**

*Let* $\gamma = \log 2 / \log \lambda$. *There exist positive constants* $C_1, C_2$ *such that for all* $s > 0$

$$C_1 \leq \lim_{t \to 0} \frac{E[|X(t)|^s]}{t^{\gamma s}} \leq C_2.$$ 

**Theorem (Laws of the iterated logarithm)**

*There exist positive constants* $C_3, C_4$ *such that*

$$C_3 \leq \limsup_{t \to 0} \frac{|X(t)|}{t^{\gamma} (\log \log (1/t))^{1-\gamma}} \leq C_4 \quad a.s.$$
Self-intersections

Theorem

If \( p_5 + p_6 + p_7 = 1 \), then, with probability 1, the scaling limit \( X \) has infinitely many self-intersections.

If \( p_7 < 1 \) and \( q_5 + \cdots + q_{10} < 1 \), then \( d_H < \log 3 / \log 2 \).

Example:

If the branching is \( w_6 \to (w_5, w_6, w_7) \to \ldots \), then following every branchings in the sets of triangles including \( y \) follows \( P_1^1 \).

Iterating this, therefore, \( X \) reaches \( y \) from opposite side of \( x \).
Conclusion

- Using the branching method, we have studied a family of self-avoiding walks on the SG.
- The limit is divided into four types in the meaning of shape.
- The case that the limit have intersections but does not fill the state space is not included in previous models.
- For the self-avoiding case, we have studied asymptotic behavior.

References
