Ollivier Ricci curvature for general graph Laplacians

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joint work with Florentin Münch (University of Postdam)
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Idea: Extend the notion of Ricci curvature introduced by Ollivier and modified by Lin-Liu-Yau to the case of general (possibly unbounded) graph Laplacians. In this setting new phenomena, which do not appear in the case of bounded operators, such as stochastic completeness can be explored.
Outline

- Framework (weighted graphs, Laplacians, curvature)
- Semigroup characterization
- Criteria for stochastic completeness
- Criteria for finiteness
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3. $|\{y \mid w(x, y) > 0\}| < \infty$ (local finiteness).
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We let

$$\text{Deg}(x) = \frac{1}{m(x)} \sum_{y \in V} w(x, y)$$

denote the weighted degree of a vertex $x$. 
We assume that $w$ is connected in the sense that for any two vertices $x$ and $y$ there exists a sequence $(x_k)_{k=0}^n$ with $x_0 = x$, $x_n = y$ and $x_k \sim x_{k+1}$ for $k = 0, 1, \ldots, n - 1$. Such a sequence is called a path connecting $x$ and $y$. 

We denote by $d(x, y)$ the usual combinatorial graph metric, that is, $d(x, y)$ is the least number of edges in a path connecting $x$ and $y$. 

We let $C(V) = \{f : V \to \mathbb{R}\}$ and let $\Delta : C(V) \to C(V)$ be given by $\Delta f(x) = \frac{1}{m(x)} \sum_{y \in V} w(x, y)(f(y) - f(x))$.

$\Delta$ is called the Laplacian associated to the weighted graph.
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Examples: (Normalized) Graph Laplacian

We give two examples based on standard edge weights and two commonly appearing measures.

**Example**

Let $w(x, y) \in \{0, 1\}$.

1. If $m = 1$, then $\Delta$ is called the **graph Laplacian** given by

$$\Delta f(x) = \sum_{y \sim x} (f(y) - f(x)).$$
Examples: (Normalized) Graph Laplacian

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Example

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1. If \( m = 1 \), then \( \Delta \) is called the graph Laplacian given by

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\Delta f(x) = \sum_{y \sim x} (f(y) - f(x)).
\]

2. If \( m(x) = d_x := \sum_{y \in X} w(x, y) = |\{y \mid y \sim x\}| \), then \( \Delta \) is called the normalized Laplacian given by

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\Delta f(x) = \frac{1}{d_x} \sum_{y \sim x} (f(y) - f(x)).
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By a direct calculation, one gets that

$$m_\varepsilon^x(y) = \begin{cases} 1 - \varepsilon \deg(x) & : y = x \\ \varepsilon \frac{w(x,y)}{m(x)} & : \text{otherwise} \end{cases}$$

where $\deg(x) = \frac{1}{m(x)} \sum_{y \in V} w(x,y)$. In particular, if $\varepsilon$ is small enough, then $m_\varepsilon^x$ is a probability measure.
For $\varepsilon > 0$, we let

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where $\text{Deg}(x) = \frac{1}{m(x)} \sum_{y \in V} w(x,y)$. In particular, if $\varepsilon$ is small enough, then $m^\varepsilon_x$ is a probability measure.

For probability measures, the transportation distance can be defined as follows:

$$W(m^\varepsilon_x, m^\varepsilon_y) = \sup_{f \in \text{Lip}(1)} \sum_{z \in V} f(z)(m^\varepsilon_x(z) - m^\varepsilon_y(z))$$

where $\text{Lip}(1) = \{ f \in C(V) \mid |f(u) - f(v)| \leq d(u,v) \}$ are the functions with Lipschitz constant 1.
A direct calculation then gives that

\[ W(m_x^\varepsilon, m_y^\varepsilon) = \sup_{f \in \text{Lip}(1)} (f(x) + \varepsilon \Delta f(x) - (f(y) + \varepsilon \Delta f(y)) \]

\[ = \sup_{f \in \text{Lip}(1)} \nabla_{xy}(f + \varepsilon \Delta f) d(x, y) \]

where \( \nabla_{xy} f := (f(x) - f(y))/d(x, y) \).
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For \( x \neq y \), let

\[ \kappa_{\varepsilon} = 1 - \frac{W(m_x^\varepsilon, m_y^\varepsilon)}{d(x, y)} \]

and define

\[ \kappa(x, y) := \lim_{\varepsilon \to 0^+} \frac{\kappa_{\varepsilon}}{\varepsilon} \]

as the *Ricci curvature* at vertices \( x \) and \( y \).
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as the \textit{Ricci curvature} at vertices \( x \) and \( y \).

Ollivier (2009) introduces this idea for Markov chains on metric spaces and looks at it for the special case of unweighted graphs for \( \varepsilon = 0 \) and \( \varepsilon = 1/2 \). Lin-Liu-Yau (2011) introduce this formula for the normalized graph Laplacian. In this case, \( \text{Deg} = 1 \).
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\Delta u(x, t) = \partial_t u(x, t) & x \in V, \ t \geq 0 \\
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We say that a graph is **Feller** if $P_t$ maps functions vanishing at infinity to functions vanishing at infinity. We let $\|\nabla f\|_\infty := \sup_{x \neq y} |\nabla_{xy} f|$ denote the Lipshitz constant of a function. We write $\text{Ric}(G) \geq K$ if $\kappa(x, y) \geq K$ for all $x, y \in V$. With these notations, we can state our first result as a characterization of lower curvature bounds for graphs which satisfy the Feller property.
Let $G$ be a weighted graph which satisfies the Feller property. The following statements are equivalent:

(i) $\text{Ric}(G) \geq K$.

(ii) For all bounded functions $f$ and all $t > 0$

$$\|\nabla P_t f\|_\infty \leq e^{-Kt} \|\nabla f\|_\infty.$$
Semigroup characterization – continued

Theorem

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This theorem gives an analogue to a result for Riemannian manifolds by Renesse and Sturm (2004). They do not assume the Feller property; however, in the manifold case a lower Ricci curvature bound immediately implies the Feller property which is not true for graphs.
Calculating the Ricci curvature

Our next result is a limit-free formula for the curvature which makes the curvature easy to calculate in some cases.

**Theorem**

Let $x \neq y$, then

$$\kappa(x, y) = \inf_{f \in \text{Lip}(1)} \nabla_{xy} \Delta f.$$
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**Theorem**

Let \( x \neq y \), then

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\kappa(x, y) = \inf_{f \in \text{Lip}(1)} \frac{\nabla_{yx} \Delta f}{\nabla_{xy} f} = 1.
\]

Idea of proof: To show that \( \frac{1}{\varepsilon} \kappa(\varepsilon, x, y) \leq \inf_{f \in \text{Lip}(1)} \nabla_{xy} \Delta f \) is easy. For the other inequality, use a cutoff argument to restrict to a compact set and create a minimizer function.
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**Example**

Line graphs Let \( V = \mathbb{N}_0 \) with \( w(m, n) = 0 \) if \( |m - n| \neq 1 \). If \( f(n) = n \) and \( r < R \), then

\[
\kappa(r, R) = \nabla_{rR} \Delta f = \frac{\Delta f(r) - \Delta f(R)}{R - r}.
\]
We now fix a reference vertex $x_0$ and let $S_r$ denote the sphere of radius $r$ about $x_0$. We let

$$\kappa(r) = \min_{y \in S_r} \max_{x \in S_{r-1}} \kappa(x, y)$$

where $x \sim y$, denote the sphere curvature for $r \geq 1$.

Our main technical tool for the results below is the following Laplace comparison theorem.

Theorem

If $f(x) = d(x, x_0)$, then

$$\Delta f(x) \leq \Delta g(x_0) - f(x) \sum_{r=1}^{\infty} \kappa(r)$$

Idea of proof: by induction on $R$ and using the formula for computing the curvature above.

Note: The inequality above is sharp on line graphs.

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Ollivier Ricci curvature
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Our main technical tool for the results below is the following Laplace comparison theorem.

**Theorem**

If $f(x) = d(x, x_0)$, then

$$\Delta f(x) \leq \text{Deg}(x_0) - \sum_{r=1}^{f(x)} \kappa(r).$$

Idea of proof: by induction on $R$ and using the formula for computing the curvature above.

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**Theorem**

*If $G$ is a weighted graph with $\kappa(r) \geq -C \log r$ for some $C > 0$ and all large $r$, then $G$ is stochastically complete.*
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Idea of proof: Use the Khas’minskii criterion for stochastic completeness along with the Laplace comparison above.
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**Theorem**

*If $G$ is a weighted graph with $\kappa(r) \geq -C \log r$ for some $C > 0$ and all large $r$, then $G$ is stochastically complete.*

Idea of proof: Use the Khas’minskii criterion for stochastic completeness along with the Laplace comparison above. Note: the result is sharp as there exist stochastically incomplete line graphs with $\kappa(r) \geq -(\log r)^{1+\varepsilon}$ for any $\varepsilon > 0$. 

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Ollivier Ricci curvature
We can also give an improved diameter bound which then implies finiteness of the graph if the weighted degree is bounded.

**Theorem**

*If there exists an $R$ with,*

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\sum_{r=1}^{R} \kappa(r) > \text{Deg}(x_0) + \max_{x \in S_R} \text{Deg}(x),
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*then $\text{diam}(G) < 2R$ where $\text{diam}(G) = \sup_{x,y \in V} d(x, y)$.*
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Idea of proof: use the Laplace comparison and argue by contradiction.
As an immediate corollary, we get the following statement.

**Corollary**

Suppose that \( \sup_{x \in V} \text{Deg}(x) < \infty \) and \( \sum_r \kappa(r) = \infty \), then the graph is finite.
As an immediate corollary, we get the following statement.

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Note: the result on the diameter is sharp.
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Suppose that \( \sup_{x \in V} \text{Deg}(x) < \infty \) and \( \sum_r \kappa(r) = \infty \), then the graph is finite.

Note: the result on the diameter is sharp. Furthermore, there exist infinite graphs with uniformly positive curvature. However, for such graph the weighted degree is unbounded.
References


Thank you for your attention!