Log-Minkowski measurability and complex dimensions

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Joint work with
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Relative fractal drum \((A, \Omega)\)

- \(\emptyset \neq A \subset \mathbb{R}^N\), \(\Omega \subset \mathbb{R}^N\), Lebesgue measurable, i.e., \(|\Omega| < \infty\)
- \(\delta\)-neighbourhood of \(A\):
  \[ A_\delta = \{ x \in \mathbb{R}^N : d(x, A) < \delta \} \]
- upper \(r\)-dimensional Minkowski content of \((A, \Omega)\):
  \[ \overline{M}^r(A, \Omega) := \limsup_{\delta \to 0^+} \frac{|A_\delta \cap \Omega|}{\delta^{N-r}} \]
- upper Minkowski dimension of \((A, \Omega)\):
  \[ \overline{\dim}_B(A, \Omega) = \inf \{ r \in \mathbb{R} : \overline{M}^r(A, \Omega) = 0 \} \]
- lower Minkowski content and dimension defined via \(\liminf\)
Minkowski measurability

\[ \dim_B(A, \Omega) = \overline{\dim}_B(A, \Omega) \Rightarrow \exists \dim_B(A, \Omega) \]

if \( \exists D \in \mathbb{R} \) such that

\[ 0 < \mathcal{M}^D(A, \Omega) = \overline{\mathcal{M}}^D(A, \Omega) < \infty, \]

we say \((A, \Omega)\) is **Minkowski measurable**; in that case

\[ D = \dim_B(A, \Omega) \]

if the above inequalities are not satisfied for \( D \), we call \((A, \Omega)\) **Minkowski degenerated**
The relative distance zeta function

- \((A, \Omega)\) RFD in \(\mathbb{R}^N\), \(s \in \mathbb{C}\) and \textbf{fix} \(\delta > 0\)

- the \textbf{distance zeta function} of \((A, \Omega)\):

\[
\zeta_{A,\Omega}(s; \delta) := \int_{A_{\delta} \cap \Omega} d(x, A)^{s-N} \, dx
\]

- dependence on \(\delta\) is not essential
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- the **complex dimensions of** $(A, \Omega)$ are defined as the poles of $\zeta_{A,\Omega}$

- take $\Omega$ to be an open neighborhood of $A$ in order to recover the classical $\zeta_A$
Holomorphicity theorem for the relative distance zeta function [LapRaŢu]

**Theorem**

- $(A, \Omega)$ RFD in $\mathbb{R}^N$:

1. $(a)$ $\zeta_{A,\Omega}(s)$ is holomorphic on $\{\text{Re } s > \dim_B(A, \Omega)\}$
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- \((A, \Omega)\) RFD in \(\mathbb{R}^N\):

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  - (b) \(\mathbb{R} \ni s < \dim_B(A, \Omega) \Rightarrow \text{the integral defining } \zeta_{A,\Omega}(s) \text{ diverges}\)
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(c) if $\exists D = \dim_B(A, \Omega) < N$ and $\mathcal{M}^D(A, \Omega) > 0$, then $\zeta_{A,\Omega}(s) \to +\infty$ when $\mathbb{R} \ni s \to D^+$
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- we call \(\{\Re s = \dim_B(A, \Omega)\}\) the **critical line**
(Generalized) complex dimensions of an RFD

**Definition**

Let $W$ be a connected open set s.t. $\{\Re s > \overline{\dim}_B(A, \Omega)\} \subset W$ and $\zeta_{A,\Omega}$ is holomorphic on $W$. The set of **visible complex dimensions of** $(A, \Omega)$ (with respect to $W$) is the set of singularities $\mathcal{P}(\zeta_{A,\Omega}, W) \subset \partial W$ of $\zeta_{A,\Omega}$. **
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**principal complex dimensions:**

$$\dim_{PC}(A, \Omega) := \{\omega \in \mathcal{P}(\zeta_{A,\Omega}, W) : \text{Re } \omega = \dim_B(A, \Omega)\}. \quad (1)$$
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- includes poles, essential and nonisolated singularities (accumulation of poles, natural boundaries)
- branching points ($W$ is then a subset of the appropriate Riemann surface) and also “mixed singularities”
An asymptotic formula for the tube function 
\[ t \mapsto |A_t \cap \Omega| \text{ as } t \to 0^+ \] in terms of \( \zeta_{A,\Omega} \).
Fractal tube formulas for relative fractal drums

- An asymptotic formula for the **tube function**
  
  \[ t \mapsto |A_t \cap \Omega| \text{ as } t \to 0^+ \]  
  
  in terms of \( \zeta_{A,\Omega} \).

**Theorem (Simplified pointwise formula with error term)**

- \( \alpha < \overline{\dim}_B(A, \Omega) < N \); \( \zeta_{A,\Omega} \) satisfies suitable rational growth conditions (**\textit{d-languidity}**) on the half-plane \( W := \{ \text{Re} s > \alpha \} \), then:

\[
|A_t \cap \Omega| = \sum_{\omega \in \mathcal{P}(\zeta_{A,\Omega}, W)} \text{res} \left( \frac{t^{N-s}}{N-s} \zeta_{A,\Omega}(s), \omega \right) + O(t^{N-\alpha}).
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- If we allow polynomial growth of \( \zeta_{A, \Omega} \), in general, we get a tube formula in the sense of Schwartz distributions.
The Minkowski measurability criterion

**Theorem (Minkowski measurability criterion)**

- $(A, \Omega)$ is such that $\exists D := \dim_B(A, \Omega)$ and $D < N$
- $\zeta_{A,\Omega}$ is \textit{d-languid} on a suitable domain $W \supset \{\text{Res } s = D\}$

Then, the following is equivalent:

(a) $(A, \Omega)$ is Minkowski measurable.

(b) $D$ is the only pole of $\zeta_{A,\Omega}$ located on the critical line $\{\text{Res } s = D\}$ and it is simple.

$$\mathcal{M}^D(A, \Omega) = \frac{\text{res}(\zeta_{A,\Omega}, D)}{N - D}$$
The Minkowski measurability criterion

- \((a) \Rightarrow (b)\): from the distributional tube formula and the **Uniqueness theorem for almost periodic distributions** due to **Schwartz**

- \((b) \Rightarrow (a)\): a consequence of a **Tauberian theorem** due to **Wiener** and **Pitt** (conditions can be considerably weakened)

- the assumption \(D < N\) can be removed by appropriately embedding the RFD in \(\mathbb{R}^{N+1}\)
an example of a **self-similar fractal spray** with a generator $G$ being an open equilateral triangle and with **scaling ratios** $r_1 = r_2 = r_3 = 1/2$

$$(A, \Omega) = (\partial G, G) \sqcup \bigsqcup_{j=1}^{3} (r_j A, r_j \Omega)$$
Fractal tube formula for The Sierpiński gasket

\[
\zeta_A(s; \delta) = \frac{6(\sqrt{3})^{1-s}2^{-s}}{s(s-1)(2^s - 3)} + 2\pi \frac{\delta^s}{s} + 3 \frac{\delta^{s-1}}{s-1}
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\[ \mathcal{P}(\zeta_A) = \{0, 1\} \cup \left( \log_2 3 + \frac{2\pi}{\log 2} \mathbb{Z} \right) \]
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|A_t| = \sum_{\omega \in \mathcal{P}(\zeta_A)} \text{res} \left( \frac{t^{2-s}}{2 - s} \zeta_A(s; \delta), \omega \right)
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\[ |A_t| = \sum_{\omega \in \mathcal{P}(\zeta_A)} \text{res} \left( \frac{t^{2-s}}{2-s} \zeta_A(s; \delta), \omega \right) \]

\[ = t^{2-\log_2 3} \frac{6\sqrt{3}}{\log 2} \sum_{k=-\infty}^{+\infty} \frac{(4\sqrt{3})^{-\omega_k}t^{-pk_i}}{(2 - \omega_k)(\omega_k - 1)\omega_k} + \left( \frac{3\sqrt{3}}{2} + \pi \right) t^2, \]

valid pointwise for all \( t \in (0, 1/2\sqrt{3}) \).
If \((A, \Omega)\) is Minkowski degenerate, \(\exists D := \dim_B(A, \Omega)\) and

\[
|A_t \cap \Omega| = t^{N-D}(F(t) + o(1)) \quad \text{as } t \to 0^+,
\]

where \(F(t) = h(t)\) or \(F(t) = 1/h(t)\) for \(h : (0, \varepsilon_0) \to (0, +\infty)\),

\(h(t) \to +\infty\) as \(t \to 0^+\) and \(h \in O(t^\beta)\) for \(\forall \beta < 0\).
Gauge Minkowski content [HeLap]

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- \(h\) is called a **gauge function of slow growth to \(+\infty\)** at \(0^+\)
- \(1/h\) is called a **gauge function of slow decay \(0\)** at \(0^+\)
- typical gauge functions: \((\log^{\circ k} t^{-1})^a\) for \(a \in \mathbb{R}^*, k \in \mathbb{N}\)
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- **\(h\)-Minkowski content:** 
\[
\mathcal{M}^D(A, \Omega, h) = \lim_{t \to 0^+} \frac{|A_t \cap \Omega|}{t^{N-D}h(t)}.
\]
The fractal nest generated by the $a$-string

$a > 0$, $a_j := j^{-a}$, $\ell_j := j^{-a} - (j + 1)^{-a}$, $\Omega := B_{a_1}(0)$

$$\zeta_{A_a,\Omega}(s) = \frac{2^{2-s}\pi}{s - 1} \sum_{j=1}^{\infty} \ell_j^{s-1}(a_j + a_{j+1})$$
Example

\[ \mathcal{P}(\zeta A_a, \Omega) \subseteq \left\{ 1, \frac{2}{a + 1}, \frac{1}{a + 1} \right\} \cup \left\{ -\frac{m}{a + 1} : m \in \mathbb{N} \right\} \]

\[ a \neq 1, \quad D := \frac{2}{1+a} \quad \Rightarrow \quad |(A_a)_t \cap \Omega| = \frac{2^{2-D}D\pi}{(2-D)(D-1)}a^{D-1}t^{2-D} + 2\pi(2\zeta(a) - 1)t \]

\[ + O(t^{2-\frac{1}{a+1}}), \quad \text{as } t \to 0^+ \]
Fractal tube formula for the fractal nest generated by the $a$-string

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$$|(A_1)_t \cap \Omega| = \operatorname{res} \left(\frac{t^{2-s}}{2-s}\zeta_{A_1, \Omega}(s), 1\right) + o(t)$$

$$= 2\pi t(-\log t) + \text{const} \cdot t + o(t) \text{ as } t \to 0^+$$
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- a pole $\omega$ of order $m$ generates terms of type $t^{N-\omega}(-\log t)^{k-1}$ for $k = 1, \ldots, m$ in the fractal tube formula
Sufficiency for log-Minkowski measurability via the Wiener-Pitt Tauberian theorem

- $m \in \mathbb{Z}$; $\zeta_{A,\Omega}^{[m]}$ denotes its the $|m|$-th derivative if $m < 0$ and the $m$-th primitive if $m > 0$;
Sufficiency for log-Minkowski measurability via the Wiener-Pitt Tauberian theorem

- $m \in \mathbb{Z}$; $\zeta_A^{[m]}$ denotes its the $|m|$-th derivative if $m < 0$ and the $m$-th primitive if $m > 0$; $\zeta_A^{[0]} := \zeta_A, \Omega$

**Theorem**

- $\overline{D} := \dim_B(A, \Omega) < N$; $\exists m \in \mathbb{Z}, \exists K > 0$, s.t. $\forall \lambda > 0$

$$G_x(y) := \zeta_A^{[m]}(x + iy) - \frac{(-1)^mK}{x + iy - \overline{D}}$$

converges in $L^1(-\lambda, \lambda)$ to a boundary function $G(y)$ as $x \to \overline{D}^+$. 


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converges in $L^1(-\lambda, \lambda)$ to a boundary function $G(y)$ as $x \to \overline{D}^+$. Then, $\exists D := \dim_B(A, \Omega) = \overline{D}$ and $(A, \Omega)$ is $h$-Minkowski measurable s.t.

$$\mathcal{M}^{D}(A, \Omega, h) = \frac{K}{N - D}, \quad (3)$$

where $h(t) := (-\log t)^m$. 
Corollary: Case of poles

Theorem

- \( \overline{D} := \dim_B(A, \Omega) < N \); \( \dim_{PC}(A, \Omega) \) consists only of poles and has no accumulation points;
Corollary: Case of poles

**Theorem**

- $\bar{D} := \dim_B(A, \Omega) < N$; $\dim_{PC}(A, \Omega)$ consists only of poles and has no accumulation points;
- $\bar{D}$ is a pole of order $m$ and all other poles on $\{\text{Re } s = \bar{D}\}$ are of order **strictly less** than $m$. 

\[ A; \Omega \rceil_{\bar{D}} \] denotes the leading coefficient of the Laurent expansion of $A; \Omega$ at $\bar{D}$. 
Corollary: Case of poles

**Theorem**

- \( \overline{D} := \dim_B(A, \Omega) < N; \dim_{PC}(A, \Omega) \) consists only of poles and has no accumulation points;
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Then, \( \exists D := \dim_B(A, \Omega) = \overline{D} \) and \( (A, \Omega) \) is \( h \)-Minkowski measurable:

\[
\mathcal{M}^D(A, \Omega, h) = \frac{\zeta_{A, \Omega}[D]_{-m}}{(N - D)(m - 1)!},
\]

where \( h(t) := (-\log t)^{m-1} \).

- \( \zeta_{A, \Omega}[D]_{-m} \) denotes the leading coefficient of the Laurent expansion of \( \zeta_{A, \Omega} \) at \( D \).
Zero-log singularities

Definition

- \( \psi, \phi \) holo. germs at \( \omega \in \mathbb{C} \) s.t. \( \omega \) is a zero of order \( m \) of \( \psi \).

We say that the holo. germ

\[
f(s) := \psi(s) \log(s - \omega) + \phi(s)
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on the principal branch of \( \log(s - \omega) \) has a zero-log singularity of order \( m \) at \( \omega \).
Zero-log singularities

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   \[ f(s) := \psi(s) \log(s - \omega) + \phi(s) \]
on the principal branch of $\log(s - \omega)$ has a zero-log singularity of order $m$ at $\omega$.

2. For instance, $f(s) = (s - 2)^3 \log(s - 2)$ has a zero-log singularity of order 3 at $\omega = 2$.
3. $\log s$ has a zero-log singularity of order 0 at $\omega = 0$, etc.
Corollary: Case of zero-log singularities

Theorem (Case of zero-log singularities)

- $\overline{D} := \overline{\dim_B}(A, \Omega) < N$; $\dim_{PC}(A, \Omega)$ consists only of zero-log singularities and has no accumulation points; 
- $\overline{D}$ is a zero-log sing. of order $m$
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- \( \overline{D} := \dim_B(A, \Omega) < N \); \( \dim_{PC}(A, \Omega) \) consists only of zero-log singularities and has no accumulation points;
- \( \overline{D} \) is a zero-log sing. of order \( m \) and all the other zero-log sings. of \( \zeta_{A,\Omega} \) on \( \{\text{Re } s = \overline{D}\} \) are of strictly higher order than \( m \).
Corollary: Case of zero-log singularities

**Theorem (Case of zero-log singularities)**

- $\bar{D} := \dim_B(A, \Omega) < N; \dim_{PC}(A, \Omega)$ consists only of zero-log singularities and has no accumulation points;
- $\bar{D}$ is a zero-log sing. of order $m$ and all the other zero-log sings. of $\zeta_{A,\Omega}$ on $\{\text{Re } s = \bar{D}\}$ are of strictly higher order than $m$.

Then, $\exists D := \dim_B(A, \Omega) = \bar{D}$ and $(A, \Omega)$ is $h$-Minkowski measurable with Minkowski content given by

$$M^D(A, \Omega, h) = (-1)^{m+1} m! \lim_{{s \to \bar{D}}} \frac{\psi(s)}{(s - \bar{D})^m},$$

(5)

where $h(t) := \frac{1}{(-\log t)^{m+1}}$. 
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Let \( f \in \text{Diff}^r(0, a) \) be continuous on \([0, a]\), positive on \((0, a)\) and let \( f(0) = f'(0) = 0 \). Assume \( 1 < x(\log(f))'(x) \). Put \( g = \text{id} - f \) and let \( S^g(x_0) = \{x_n| n \in \mathbb{N}\} \) be an orbit of \( g \), \( x_0 < a \).

\[
|A_t(S^g(x_0))| \asymp g^{-1}(t)
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Let $f \in \text{Diff}^r(0, a)$ be continuous on $[0, a)$, positive on $(0, a)$ and let $f(0) = f'(0) = 0$. Assume $1 < \alpha(\log(f))'(x)$. Put $g = \text{id} - f$ and let $S^g(x_0) = \{x_n| n \in \mathbb{N}\}$ be an orbit of $g$, $x_0 < a$.

$$|A_t(S^g(x_0))| \asymp g^{-1}(t)$$

- Let $g(x) = x^k \log^\alpha(x^{-1})$, then $g^{-1}(t) \asymp \frac{t^{1/k}}{\log^\alpha(t^{-1})^{1/k}}$
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- Let $g(x) = x^k \log^\circ m(x^{-1})$, then $g^{-1}(t) \asymp \frac{t^{1/k}}{\log^\circ m(t^{-1})^{1/k}}$
- we also have $|A_t(S^g(x_0))| \asymp t(- \log t)$ for appropriate differentiable $f$
The 1/2-square fractal

**Figure:** Here, $G := G_1 \cup G_2$ is the single generator of the corresponding self-similar spray or RFD $(A, \Omega)$, where $\Omega = (0, 1)^2$. 
Fractal tube formula for the $1/2$-square fractal

$$\zeta_A(s) = \frac{2^{-s}}{s(s - 1)(2^s - 2)} + \frac{4}{s - 1} + \frac{2\pi}{s}, \quad (6)$$

$$D(\zeta_A) = 1, \quad \mathcal{P}(\zeta_A) := \mathcal{P}(\zeta_A, \mathbb{C}) = \{0\} \cup \left(1 + \frac{2\pi}{\log 2} \mathbb{i}\mathbb{Z}\right). \quad (7)$$
Fractal tube formula for the $\frac{1}{2}$-square fractal

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(7)

$$|A_t| = \sum_{\omega \in \mathcal{P}(\zeta_A)} \operatorname{res}\left(\frac{t^{2-s}}{2-s} \zeta_A(s), \omega\right)$$

$$= \frac{1}{4 \log 2} t \log t^{-1} + t \ G(\log_2(4t)^{-1}) + \frac{1 + 2\pi}{2} t^2,$$

(8)

valid for all $t \in (0, \frac{1}{2})$, where $G$ is a nonconstant 1-periodic function on $\mathbb{R}$ bounded away from zero and $\infty$.

The $\frac{1}{2}$-square fractal is critically fractal in dimension 1.
Further research directions

- Riemann surfaces generated by relative fractal drums
- Extending the notion of complex dimensions to include complicated “mixed” singularities/branching points and connecting them with various gauge functions
- Obtaining corresponding tube formulas and gauge-Minkowski measurability criteria
- Applying the theory to problems from dynamical systems

