The Homotopy Critical Spectrum for Non-Geodesic Spaces

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Fractals 6, Cornell University

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Outline

1. Discrete Homotopy Theory
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2. Applications for Geodesic Spaces
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3. Issues for Non-geodesic Metric Spaces
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4. Chained Metric Spaces
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3. Issues for Non-geodesic Metric Spaces
4. Chained Metric Spaces
5. Elaboration will be for informal discussion of resistance metrics
A little history

- 2001 Berestovskii-P.: generalized covering spaces of topological groups based on a construction of Schreier from the 1920’s, rediscovered by Malcev in the 1940’s, reinterpreted by us in terms of discrete chains and homotopies

- 2001: Sormani-Wei independently developed the idea of $\delta$-covers of geodesic spaces, using a construction of Spanier

- 2007 Berestovskii-P. extended discrete homotopy ideas to uniform spaces (hence metric spaces)

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Let $X$ be a metric space.

**Definition**
For $\varepsilon > 0$, an $\varepsilon$-chain is a finite sequence $\{x_0, ..., x_n\}$ such that for all $i$, $d(x_i, x_{i+1}) < \varepsilon$. 
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An $\varepsilon$-homotopy consists of a finite sequence $\langle \gamma_0, ..., \gamma_n \rangle$ of $\varepsilon$-chains, where each $\gamma_i$ differs from its predecessor by a “basic move”: adding or removing a *single* point, always leaving the endpoints fixed.
Definition
Fixing a basepoint \(*\), \(X_\varepsilon\) is defined to be the set of all \(\varepsilon\)-homotopy equivalence classes of \(\varepsilon\)-chains starting at \(*\), and \(\phi_\varepsilon : X_\varepsilon \rightarrow X\) is the endpoint map. Equivalence classes are denoted by \([\alpha]_\varepsilon\).
Epsilon-Covers

Definition
Fixing a basepoint *, $X_\varepsilon$ is defined to be the set of all $\varepsilon$-homotopy equivalence classes of $\varepsilon$-chains starting at *, and $\phi_\varepsilon : X_\varepsilon \to X$ is the endpoint map. Equivalence classes are denoted by $[\alpha]_\varepsilon$.

Definition
The group $\pi_\varepsilon(X)$ is the subset of $X_\varepsilon$ consisting of classes of $\varepsilon$-loops starting and ending at * with operation induced by concatenation, i.e., $[\alpha]_\varepsilon \ast [\beta]_\varepsilon = [\alpha \ast \beta]_\varepsilon$. 
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- $\phi_\varepsilon : X_\varepsilon \rightarrow X$ is an isometry from any $\frac{\varepsilon}{2}$-ball onto its image
Homotopy Critical Values

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An $\varepsilon$-loop $\lambda$ in a metric space $X$ is called $\varepsilon$-critical if $\lambda$ is not $\varepsilon$-null, but is $\delta$-null for all $\delta > \varepsilon$. When an $\varepsilon$-critical $\varepsilon$-loop exists, $\varepsilon$ is called a homotopy critical value; the collection of these values is called the Homotopy Critical Spectrum.
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- Homotopy critical values are determined by lengths of “essential circles”, which are very special closed geodesics.
- Therefore the homotopy critical spectrum corresponds to a subset of the length spectrum and differs from the Sormani-Wei “covering spectrum” by a factor of $\frac{2}{3}$. 
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- That is, there exist isospectral manifolds with different covering spectra.
Generalizations of Gromov, Anderson, Shen-Wei Theorems

Theorem

(P.-Wilkins) Suppose $X$ is a semilocally simply connected, compact geodesic space of diameter $D$, and let $\varepsilon > 0$. Then for any choice of basepoint, $\pi_1(X)$ has a set of generators $g_1, \ldots, g_k$ of length at most $2D$ and relations of the form $g_i g_m = g_j$ with

$$k \leq \frac{8(D + \varepsilon)}{\varepsilon} \cdot \Gamma(X, \varepsilon) \cdot C \left( X, \frac{\varepsilon}{4} \right)^{\frac{8(D + \varepsilon)}{\varepsilon}}.$$
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**Corollary**

Let $\mathcal{X}$ be any Gromov-Hausdorff precompact class of semilocally simply connected compact geodesic spaces. If there are numbers $\varepsilon > 0$ and $N$ such that for every $X \in \mathcal{X}$, $\Gamma(X, \varepsilon) \leq N$, then there are finitely many possible fundamental groups for spaces in $\mathcal{X}$. 
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Jay Wilkins showed that there are metric spaces whose homotopy critical spectrum is $[0, 1]$. 
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Refinement is the Key

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These concepts fail for the “bad” examples above.
Chained Metric Spaces

**Definition**
A metric space $X$ is called “chained” if whenever $d(x, y) < \varepsilon$ and $0 < \delta < \varepsilon$, then $x$ and $y$ can be joined by a $\delta$-chain that lies entirely in $B(x, \varepsilon) \cap B(y, \varepsilon)$. Equivalently, $B(x, \varepsilon) \cap B(y, \varepsilon)$ is “chain connected”.

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- If $(X, d)$ is a chained metric space and $f : [0, \infty) \rightarrow [0, \infty)$ is a concave increasing function such that $f(0) = 0$ then $(X, f \circ d)$ is chained.
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- So for example, if $(X, d)$ is a geodesic metric then $(X, d^{\frac{1}{2}})$ has no rectifiable curves but is still chained.
- A stronger (and more geometrically appealing) condition is: Every $x, y \in X$ may be joined by a curve $c : [0, 1] \to X$ such that $d(x, c(t))$ is increasing and $d(y, c(t))$ is decreasing.
Finiteness

**Theorem**

Let $X$ be a compact chained metric space and $\varepsilon > 0$. Then there are at most

$$2C(X, \frac{\varepsilon}{4})^{40C(X, \frac{\varepsilon}{2})}$$

homotopy critical values $\delta$ such that $\delta \geq \varepsilon$. In particular, the homotopy critical spectrum is discrete in $(0, \infty)$. Moreover, this number is uniformly bounded in any Gromov-Hausdorff precompact class.
Thank You