Lipschitz equivalence of self-similar sets and hyperbolic graphs

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Lipschitz Equivalence

\( X \simeq Y \) metric spaces, \( \exists \) bi-Lipschitz map \( \sigma : X \to Y \):

\[ C^{-1}d(x, y) \leq d(\sigma(x), \sigma(y)) \leq Cd(x, y). \]

This implies \( \dim_H X = \dim_H Y \), but the converse is not true

Poineers: Cooper & Pignartaro (88), Falconer & Marsh (89)
On self-similar set with the strong separation condition, and the algebraic relationship of the contraction ratio
Are the following two totally disconnected self-similar sets Lipschitz equivalent? \( \{1, 3, 5\} \) – \( \{1, 4, 5\} \) problem (David and Semmes):

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\]


- Many developments on totally disconnected self-similar/conformal sets: Xi-Xiong, Ruan-Rao-Wang, Deng-He, Llorente-Mattila, etc.

- We make use of the approach of augmented tree structure on the symbolic space to study the problem.

- Augmented tree was introduced by Kaimanovich 03 in connection with hyperbolic graphs.
Recall \( \{S_j\}_{j=1}^N \) an IFS of similitudes with contractive ratio \( 0 < r < 1 \), the self-similar set is \( K \).

Symbolic space \( \Sigma^* = \bigcup_{n=0}^\infty \Sigma^n \) where \( \Sigma = \{1, \cdots, N\} \). It has a tree structure. Let

\[
\rho(u, v) = r^{\min\{k: i_{k+1} \neq j_{k+1}\}}.
\]

- The completion of \((\Sigma^*, \rho)\) is \( \Sigma^* \cup \Sigma^\infty \) and is compact.
- \( \Sigma^\infty \) is an \( N \)-Cantor set. We can consider \( \Sigma^\infty \) as a boundary of \((\Sigma^*, \rho)\).
- Let \( \pi: \Sigma^\infty \to K \) be the natural projection. Then \( \Sigma^\infty / \sim\pi \) is a homeomorphic to \( K \) (here \( u \sim\pi v \) if \( \pi(u) = \pi(v) \)).
In the 80’s, Gromov developed a simple and elegant theory of hyperbolic graph. Many of the geometric, analytic, dynamic and stochastic results on the Poincaré disc can be extended to hyperbolic groups.
Graph: vertices, edges, path, geodesic, tree

Hyperbolic graph $X$ (Gromov): *Any geodesic triangle in $X$ is $\delta$-thin*, i.e., $\exists \delta > 0 \ni \exists$ any point $x$ in the geodesic triangle has distance $\leq \delta$ to the other two sides.

- A tree is a hyperbolic graph with $\delta = 0$
The graph of the binary tree together with the horizontal edges. It is an analog of the hyperbolic upper half-plane.

Actually it is the $\Sigma^*$ for $S_1(x) = \frac{1}{2}(x)$, $S_2(x) = \frac{1}{2}(x + 1)$ together with the horizontal edges joint $u, \nu$:

$$S_u([0, 1]) \cap S_\nu([0, 1]) \neq \emptyset, \quad u, \nu \in \Sigma^n$$
Gromov product

$$(u|v) := \frac{1}{2}(|u| + |v| - d(u, v))$$

where $|u|$ is distance from the root $\vartheta$ to $u$, $d(u, v)$ is the geodesic distance. Then

$$d(\vartheta, \pi(u, v)) - 2\delta \leq (u|v) \leq d(\vartheta, \pi(u, v))$$

($\pi(u, v)$ is a geodesic from $u$ to $v$)

Define

$$\rho_a(u, v) = \begin{cases} e^{-a(u|v)}, & u \neq v, \\ 0, & u = v. \end{cases}$$

Let $\hat{X}$ be the completion of $(X, \rho_a)$. The hyperbolic boundary is

$$\partial X = \hat{X} \setminus X$$

- Compare this with the tree of symbolic space $\Sigma^*$. 
Augmented tree on $\Sigma^*$ (Kaimanovich 03)

Let $\mathcal{E}_v$ denote the edges of the tree. Define

$$\mathcal{E}_h = \{(u, v) \in \Sigma^n : S_u(K) \cap S_v(k) \neq \emptyset, n \geq 1\}$$

and let $\mathcal{E} = \mathcal{E}_v \cup \mathcal{E}_v$, we call $(X, \mathcal{E})$ an augmented tree.

**Theorem 1** (Wang and L, Ind.U. 09) Suppose $\{S_j\}_{j=1}^N$ satisfies the OSC. Then the augmented tree is a hyperbolic graph.
Identifying $K$ with the hyperbolic boundary

Define $\Phi : (\partial \Sigma^*) \longrightarrow K$ by $\Phi(\xi) = \lim_{n \to \infty} S_{i_1 \ldots i_n}(x_0)$.

**Theorem 2.** (Wang and L, 09) $\{S_j\}_{j=1}^N$ with OSC + ..., then

$$C^{-1}|\Phi(\xi) - \Phi(\eta)| \leq \rho_a(\xi, \eta)^\alpha \leq C|\Phi(\xi) - \Phi(\eta)|.$$

*where $\alpha = \log r/a$.*

- The OSC can be weaken considerably and the theorems still hold (Wang Math.Z. 14)

Let $X, Y$ be two hyperbolic graphs, a map $\sigma$ is called a near-isometry of $X$ and $Y$ if $\exists$ finite set $E \subset X, F \subset Y \ni \sigma : X \setminus E \to Y \setminus F$ is a bijection, and

$$|d(\sigma(x), \sigma(y)) - d(x, y)| \leq c$$

**Proposition.** Suppose $X, Y$ are hyperbolic augmented trees, and there exists a near-isometry between them, the $\partial X \simeq \partial Y$. 
Let $X$ be an $N$-ary tree with an augmented tree structure. Call $T$ a horizontal component if $T$ is a maximal connected subset in $X_n = \{ u : |u| = n \}$. Let

$$T_D = \{ u \mathbf{i} : u \in T, \mathbf{i} \in \Sigma^* \}.$$

Two components $T$ and $T'$ are equivalent if $T_D$ and $T'_D$ are graph isomorphic.

$(X, \mathcal{E})$ is called simple if it has only finitely many equivalent classes of horizontal components.

**Proposition.** Suppose $(X, \mathcal{E})$ is simple, then the augmented tree is hyperbolic, and $K$ is totally disconnected.
**Theorem 3.** (Main Theorem) Suppose $X$ is an $N$-ary augmented tree and is simple, then there is a near-isometry $\sigma : (X, \mathcal{E}) \to (X, \mathcal{E}_v)$. Hence $(\partial X, \mathcal{E}) \sim (\partial X, \mathcal{E}_v)$. ($(\partial X, \mathcal{E}_v)$ is an $N$-Cantor set.)

**Theorem 4.** Suppose $\{S_j\}_{j=1}^N$ has same contraction ratio $+ \ldots$, and the augmented tree is simple. Then $K$ is Lipschitz equivalent to an $N$-Cantor set.

\[
\begin{align*}
(\partial \Sigma^*, \mathcal{E}) & \xrightarrow{Th3} (\partial \Sigma^*, \mathcal{E}_v) \\
K & \xrightarrow{Th2} \text{CantorSet}
\end{align*}
\]

The theorem can also be extended to certain finite union of copies of itself.
Main idea of proof of Theorem 3.

• Suppose \((X, \mathcal{E})\) is simple. Let \([T_1], \ldots, [T_k]\) be the connected components; 
  \(Z_{i1}, \ldots Z_{i\ell}\) the connected components of \(T_i\Sigma\) (the descendants of \(T_i\));
  \(a_{ij}\) the number of \(Z_{i1}, \ldots, Z_{i\ell}\) belongs to \([T_j]\). Define incidence matrix 
  \[A = [a_{ij}]\]

• Let \(b = (b_1, \ldots, b_r)\) be the length of each \(T_i\), then \(Ab = Nb\)

• If \(A\) is primitive (i.e., \(A^k > 0\)), then \(A^\ell\) is \((N^\ell, b)\)-rearrangeable.
The concept of rearrangement (Deng-He, 12)

Consider a class of connected components, each has size $b_i$ and there are $a_i$ of them. Let $a = (a_1, \cdots, a_r)$, $b = (b_1, \cdots, b_r)^t$. There is a total $ab := Np$ units.

We say that $a$ is $(N, b)$-rearrangeable if we can divide this class into $p$ groups, each group has $N$ units.

Lemma For $\min_i\{a_i\} \gg \max_j\{b_j\}$, and $\gcd(b)$ divides $N$, then $a$ is $(N, b)$-rearrangeable.

• A nonnegative integer matrix $A$ is called rearrangeable if every row of $A$ is rearrangeable.
• In the case $A$ is primitive and $Ab = Nb$, $A$ (in fact $A^\ell$) is rearrangeable. Then there is $\sigma : (\Sigma^*, E) \rightarrow (\Sigma^*, E_v)$ satisfies

$$|d(\sigma(u), \sigma(v)) - d(u, v)| \leq C \quad \forall u, v \in \Sigma^*$$

(near isometry). $\sigma$ is a bi-Lipschitz map with respect to the hyperbolic metric.

• It induces a bi-Lipschitz map $\sigma : (\partial \Sigma^*, E, \rho_a) \rightarrow (\partial \Sigma^*, E_v, \rho_a')$
• If $A$ is not primitive, then there exists $\ell$ such that

$$P^t A^\ell P = \begin{bmatrix} A_1 & * \\ \vdots & \ddots \\ 0 & A_k \end{bmatrix}$$

where $A_i$ are either zero or primitive ($P$ rotation of the basis).

• Use the rearrangeable property for the $T_i$’s in $A_k$ to obtain a near-isometry $\sigma_k$ on the subgraphs $(T_i)_D$ as before.

• Define a new notion of quasi-rearrangeable for the the $A_i$. Then use it for the $T_i$ in $A_{k-1}$ together with the $\sigma_k$ to obtain $\sigma_{k-1}$ on the $T_i$’s of the two matrices. Then inductively.
Example 1. The $\{1, 3, 5\} - \{1, 4, 5\}$ case, $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Example 2. Modify the $\{1, 4, 5\}$ by taking $S_2$ to have a reflection: $S_2(x) = \frac{1}{5}(-x + 3)$. Then $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$, and $K \simeq 3$-Cantor set.

• Note that in the first case $A$ is primitive, but not the second one.
**Example 3.** $K \simeq 4$-Cantor set.

**Example 4.** $K = K_1 \cup K_2 \simeq 3$-Cantor set

**Proposition.** $C = 3$-Cantor set in $\mathbb{R}$. Then $C \cup (C + \alpha) \simeq C$ if $\alpha > 1$ or $0 \leq \alpha \leq 1$ is a rational.
Remarks and further work:

1. For the augmented edges

\[ E_h = \{(u, v) \in \Sigma^n : S_u(K) \cap S_v(k) \neq \emptyset, n \geq 1\}, \]

the defining condition can be replaced by other conditions and still get a hyperbolic graph.

2. It is possible to replace the main condition of ”simple augmented tree” to ”the length of connected horizontal component is uniformly bounded”.

3. It’s likely to extension the contraction ratio to be exponentially commensurable (early work by Falconer-Marsh (89) and recent work by Xi-Xiong)
Question: How to handle Lipschitz equivalence for the non-totally disconnected cases?

(Luo-Rao-L, Cambridge, 13)
Thank You