Manifestations of the Lamplighter

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May 6-th, Cornell, Topological Fest
The main HERO

$L = \mathbb{Z}_2 \ast \mathbb{Z}$ - Lamplighter

$\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$

His "command"

$F \leq G$, $|F| < \infty \Rightarrow |G| = \infty$

Other personages: iterated wreath products.
I. The permutational wreath product.

\[ A, G \text{ two groups, } A \neq \{ e \} \]

\[ G \wr X \text{ G acts on } X \]

\[ W = \left( \prod_x^* A \right) \rtimes G = A \wr G - \text{ restricted wreath product} \]

\[ \prod_x^* A \text{ - weak direct product (only finitely many entries are } \neq \{ e \} \text{)} \]

\[ G \text{ act on } \prod_x^* A \text{ on the left by } (g \cdot f)(x) = f(xy) \]
\[ A \wr G = (\prod A) \times G \quad \text{unrestricted wreath product} \]

\[ W_r = \bar{Z} \quad \quad \text{wr} = \mathbb{Z} \]

\[ |A|, |B| < \infty \implies A \wr B = A \bar{Z} B \]

**Examples:**

\[ \mathcal{L} = \mathbb{Z} \wr \mathbb{Z} \quad \text{The Lamplighter} \]

\[ |F| \geq 2, \quad G = \mathbb{Z}^d, \quad F_d, \quad B(m,n) \text{-free Burnside gp, group of intermediate growth} \]

\[ \mathbb{Z} \wr \mathbb{Z}, \quad \text{iterated wreath products} \]
$A, G, S_A, S_G$ - systems of generators for $A$ and $G$

$S = S_A \cup S_G$ - system of generators for $A \ast G$

$G \ni g \rightarrow (1, g)$

$A \ni a \rightarrow (f_a, 1)$

$\text{embeddings}$

$f_a(x_0) = a, \quad f_a(x) = e, \quad x \neq x_0, \quad x_0 \in X$ - distinguished point

$A$ and $B$ are finitely generated $\Rightarrow$ $A \ast B$ is f.g.

G. Baumslag. $A \ast B$ is not finitely presentable if $A \neq \{e\}$ and $|B| = \infty$
the Lamplighter is not finitely presentable

\[ L = \langle a, b \mid a^2 = 1, [a, a^b]^n = 1, n = 1, 2, \ldots \rangle \]

\[ x^y = y^{-1}x y \quad [x, y] = x^{-1}y^{-1}xy \]

Th. [Baumslag, Remeslennikov] Every finitely generated metabelian group embeds into a finitely presented metabelian group.

Apply this to the Lamplighter.
\[ a = (0, 0, 0, \ldots) \in \bigoplus \mathbb{Z}_2 \quad \text{— base group} \]

\[ b \quad \text{— a generator of active group } \mathbb{Z}. \]

Th. [Baumslag]. Let \( \alpha: \mathbb{Z} \to \mathbb{Z} \) be given by \( \alpha(a) = [a, b], \) \( \alpha(b) = b. \) This defines an injective group homomorphism, and

\[ G = \langle a, b, s \mid a^2 = [b, s] = [b^2 a b, a] = e, \quad s^{-1} a s = [a, b] \rangle \]

is isomorphic to the ascending HNN extension of \( \mathbb{Z} \) along \( \alpha. \)

\( \mathbb{Z} \hookrightarrow G \) is Baumslag–Remeslennikov type embedding

[will be used for Atiyah Problem].
Growth

$G$ - finitely generated group

$A$ - finite system of generators

$|g| = |g|_A$ - the length of $g$ w.r.t. $A$

$\delta(n) = \delta^A_G(n) = \# \{ g \in G \mid |g| \leq n \}$

$= \# B_e(n)$ - ball of radius $n$ in Cayley graph
\[ \gamma_G^A(n) \sim \gamma_G^B(n) \]

Minor equivalence

\[ \sum_{\gamma(n)} \rightarrow \text{the growth degree of } G \]

\[ \Gamma_G(2) = \Gamma_G^A(2) = \sum_{n=0}^{\infty} \gamma(n) 2^n \rightarrow \text{growth series} \]

\[ \Gamma(2) \text{ rational} \Rightarrow \gamma(n) \sim P(n) \text{ or } \gamma(n) \sim 2^n \]

or even algebraic

\[ \uparrow \text{polynomial} \]

\[ \uparrow \text{exponential} \]
Th. [W. Parry 92]. For $G = F_2 F_m$, $m \geq 2$ the growth series is an algebraic irrational function.
Growth can be polynomial, intermediate (between polynomial or exponential), and exponential.

Th. (GRe). 1) There are uncountably many 2-generated groups of intermediate growth.

2) The partially ordered set of growth degrees contains a chain of the cardinality of the continuum and contains an antichain of the cardinality of the continuum.

Corollary. Up to quasi-isometry there are uncountably many 2-generated groups.
The main example. \( G = \langle a, b, c, d \rangle \) \( G_X = \{0, 1\} \setminus \mathbb{Q}_2 \) diadic rational points

\[ a \]
\[ b \]
\[ c \]
\[ d \]

- permutation of two halves of the interval
- identity transformation

**Th. (Gri. 84)**

\[ e^{\frac{\ln n}{n}} \leq \gamma(n) \leq e^{\frac{\beta n}{n}}, \quad \beta = 0.52^{31} < 1 \]

Y. Leonov, D. Bartholdi
\[ \frac{1}{2} \Rightarrow \frac{1}{2} + 0.4, \quad \beta \Rightarrow 0.74 \ldots \]

Bartholdi, Muchnik, Pat
**General construction**

\[ \mathcal{Q} = \{0, 1, 2\} \Rightarrow \omega = \omega_1 \omega_2 \ldots \quad - \text{infinite sequences} \]

\[ 0 \leftrightarrow \begin{pmatrix} p \\ p \\ 1 \end{pmatrix} \quad 1 \leftrightarrow \begin{pmatrix} p \\ 1 \\ p \end{pmatrix} \quad 2 \leftrightarrow \begin{pmatrix} 1 \\ p \\ p \end{pmatrix} \]

\[ \omega \leftrightarrow \begin{pmatrix} U_1^\omega \\ V_1^\omega \\ W_1^\omega \end{pmatrix} = \begin{pmatrix} u_1^\omega & w_1^\omega & v_1^\omega \\ u_2^\omega & w_2^\omega & v_2^\omega \\ \vdots & \vdots & \vdots \end{pmatrix}_{3 \times \infty} \]

\[ b_\omega = \begin{pmatrix} u_1^\omega \\ u_2^\omega \end{pmatrix} \quad c_\omega = \begin{pmatrix} v_1^\omega \\ v_2^\omega \end{pmatrix} \quad d_\omega = \begin{pmatrix} w_1^\omega \\ w_2^\omega \end{pmatrix} \]

\[ G_\omega = \langle a, b_\omega, c_\omega, d_\omega \rangle \]

\[ G = \langle a, b, c, d \rangle = G_{(012)^\infty} \]

\[ G_{(01)^\infty} \quad - \text{also important gp} \]
\[ \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \]

\[ \mathcal{L}_2 = \{ \omega \mid \omega \text{ is constant at } \infty \} \]

\[ \omega = \omega_1, \ldots, \omega_n, i^i \ldots i^i, i \in \{1, 2, 3\} \]

1. \( \omega \in \mathcal{L}_1 \Rightarrow G_\omega \text{ is of intermediate growth and is branch just-infinite group.} \)

2. \( \omega \in \mathcal{L}_2 \Rightarrow G_\omega \text{ is virtually abelian group.} \)

The "surgery" of the construction: Delete from \( \{ G_\omega \}_{\omega \in \mathcal{L}_2} \) the countable set \( \{ G_\omega \}_{\omega \in \mathcal{L}_2} \) and take the closure in the space of marked 4-generated groups.
We replace $G_0$, $w \in \Omega_2$ by $\tilde{G}_0$ - virtually metabelian groups of exponential growth, and get a Cantor set of groups.

$$\tilde{G}_0^\infty \cong \mathbb{Z} \times \mathbb{Z}_2 \cong G(A_{891})$$

The presence of the Lamplighter in this construction allowed to get uncountably many growth degrees.

$$e^{n^{1/2+0.04}} \leq \gamma G(0,2)^{\omega^m}(n) \leq e^{n^p}$$

$$V > 0, \quad e^{\frac{p}{\log^2 n}} \leq \gamma \leq e^{\frac{p}{\log^4 n}}$$


Q. Is $X G(0,1)^{\omega^m}(n) \sim e^{n^{1/3+0.04}}$ for some $s$? What is $s$?

Q. Is $\gamma G(0,1)^{\omega^m}(n) \sim e^{n^{\frac{1}{\log^5 n}}}$ for some $s$, $1 \leq s \leq 2$? What is $s$?
There are two infinite sequences of groups \( \{G_k\}_{k=1}^{\infty} \) and \( \{H_k\}_{k=1}^{\infty} \) and a sequence of positive numbers \( \alpha_k = 1 - (1 - \alpha)^k \) where \( \alpha = \log(2) / \log(\frac{5}{3}) \approx 0.764 \). and \( \eta \) is a root of \( x^3 + x^2 + x - 2 \) s.t.

\[
\begin{align*}
\varphi_{G_k}(h) &\sim e^{h \alpha_k} \\
\varphi_{H_k}(h) &\sim e^{(\log h) \eta \alpha}
\end{align*}
\]

The construction is based on the use of permutational wreath product and the notion of \textit{inverted orbit growth}. 

\[L. Bartholdi \ and \ A. Erschler 2010\]
\[ G = \langle a, b, c, d \rangle = G_{0,1,2,\ldots} \text{ act on a binary rooted tree} \]

\[ P = \text{St}_G(1^\infty) \text{ - stabilizer} \]

\[ \Gamma = \Gamma(G, P, \{a, b, c, d\}) \text{- Schreier graph} \]

\[ \gamma \in \partial \Gamma = \{0, 1, 2, \ldots\} \text{ - boundary} \]

\[ \Gamma \text{ has linear growth } \sim n \text{ but sublinear inverted growth of the type } n^{1-\alpha}, \quad \alpha = 0.764... \]
$G \times X$ (right action) $X \ni x_0$. A distinguished point $X \ni x_0$.

Inverted orbit of a word $w = g_1 \cdots g_n$ over $G$ is

\[ \{ x_0 g_1 \cdots g_n, x_0 g_2 \cdots g_n, \ldots, x_0 g_n \cdots g_n \} = IO_{x_0}(w) \]

\[ \delta(w) = \# IO_{x_0}(w) \]

Inverted orbit growth function is

\[ \Delta(n) = \max \{ \delta(w) \mid |w| = n \} \]

Th. Let $A$ be a non-trivial group having growth $\sim e^h$, and assume $\frac{h}{p(n)}$ is convex. Consider the wreath product $M = A \wr \mathbb{Z}$. Then the growth of $M$ is $\sim e^{\frac{h}{p(h^{1-\alpha})}}$. Bartholdi, Erschler.
Amenability

J. von Neumann (discrete case) 1929

L. Alphors (amenable Riemannian surfaces) 1935

N. N. Bogolyubov (general topological groups) 1939

Def. [A group is amenable if it does not allow Ponzi schemes]

1) $G$ is amenable if it has $\text{LIM}$ (left invariant mean) i.e. a finitely additive left invariant measure $\mu$ defined on the $\sigma$-algebra of all subsets of $G$ with values in $[0,1]$ and normalized by $\mu(G)=1$. 
2) A group $G$ is superamenable if for every action $G \times X$ and nonempty subset $A \subset X$ there is a finitely additive measure $\mu$ defined on the $\sigma$-algebra of all subsets of $X$ with values in $[0, \infty]$ normalized by the condition $\mu(A) = 1$.

Superamenable $\Rightarrow$ amenable $\Rightarrow$ superamenable.

$F_2 \hookrightarrow G \Rightarrow G$ is non amenable.

Free group

$FS_2 \hookrightarrow G \Rightarrow G$ is not superamenable.

$G$ is amenable $\iff$ $\forall$ action on a compact space $X$ there is a $G$-invariant probabilistic measure on $X$ (Bogolyubov 1939)
G is superamenable ⇔ ∀ action on a topological space there is an invariant Radon measure.

J. Rosenblatt introduced the notion of superamenable $f.p.$, proved that subexponential growth $\Rightarrow$ superamenability and conjectured. A group is superamenable if and only if it is amenable and does not contain a free semigroup $FS_2$ on two generators.

Grigorchuk in 1987 gave a counter-example.

Theorem: Let $G = \langle a, b, c, d \rangle$ be a 2-group of intermediate growth and $L = \mathbb{Z}_2 \ast G$. Then $L$ is torsion amenable group (and hence does not contain $FS_2$) but is not superamenable.
It was showed that instead of \( S^2 \) \( \mathbb{Z} \) contains a paradoxical binary rooted tree.

Two related problems of Rosenblatt.

Problem. Does any group \( G \) of exponential growth admit a Lipschitz imbedding of the infinite binary tree?

Problem. Is every supramenable group exponentially bounded? (i.e. of subexponential growth).
Föllner criterion and Föllner function

A finitely generated group $G$ is amenable $\iff$

$$\inf_{E \in \mathcal{V}(\Gamma)} \frac{|\partial E|}{|E|} = 0$$

$\Gamma = \Gamma(G, A)$ — Cayley graph, $\partial E$ — boundary of a subset

$$F(r) = F_{\delta l}(r) = \min \{ |E| : E \in \mathcal{V}(\Gamma), \frac{|\partial E|}{|E|} < \frac{1}{r} \}$$

$r \in (1, +\infty)$

A. Vershik, 70th.

Q. How $F(r)$ grow as $r \to +\infty$?
Varopoulos, Coulhon and Saloff-Coste, Pittet and Saloff-Coste:

1) $G$ is virtually nilpotent with polynomial growth of type $n^d$, then $F(r) \sim r^d$.

2) F"olner function of $\mathbb{Z} \ltimes \mathbb{Z}^d$, $d \geq 2$ is super-exponential

Th. [A. Erschler]. There exists $C > 0$ such that the following holds. Let $A$ and $B$ be two finitely generated amenable groups ($|A| \geq 2$). Let $S_A$ and $S_B$ be finite generating sets of $A$ and $B$ respectively. Then
\[ F_{A \rightarrow B}(r) \geq C \left( F_A(Cr) \right)^{F_B(Cr)} \]

\[ \Rightarrow F_{\mathbb{Z}^2 \mathbb{Z}}(r) \sim r^r \quad F_{\mathbb{Z}_k \mathbb{Z}}(r) \sim e^{rd} \quad \text{super exponential growth} \]

- \( F \leq G \), \( F \) finite
- \( G \) of polynomial growth \( d \)
- \( H \leq G \), \( H \) infinite and of polynomial growth, \( G \) of polynomial growth \( d \)
- \( F \leq (\ldots (F_2(F_2\mathbb{Z}))\ldots) \)
- \( F \leq \ldots \), \( k \)-times iterated wreath product

\[ e^r = \exp_k(r) \]
\( \mathbb{Z} \times (\mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z})) \ldots \)

\( k \) times iterated wreath product, \( k \geq 2 \)

\( \exp(k) (r \log r) \)
The Dixmier Problem

$G$ is said to be unitalisable if every uniformly bounded representation $\pi: G \to \mathcal{B}(H)$ is unitarisable.

i.e. there is an invertible operator $S$ on $H$ s.t. $S \pi(\cdot) S^{-1}$ is a unitary representation.

**Dixmier 1950**: amenable $\Rightarrow$ unitarizable

$\Leftarrow ?$ - Dixmier Problem.

$F_0 \hookrightarrow G \Rightarrow G$ is not unitarizable

G. Pisier
D. Osin, N. Monod and N. Ozawa produced examples of non-unitarisable groups without free subgroups.

Th. [Monod and Ozawa] For any group $G$, the following assertions are equivalent.

(i) The group $G$ is amenable.

(ii) The wreath product $A \wr G$ is unitarizable for all abelian groups $A$.

(iii) The wreath product $A \wr G$ is unitarizable for some infinite group $A$.

$\Rightarrow$ Th. [Monod & Ozawa] The free Burnside group $B(m, np)$, $m, n \geq 2$, $p \geq 65$, and $p$ odd of exponent $np$ is non-unitarizable.
Random walks

$G$, $\mu$-symmetric measure on $G$, $\mu(g) = \mu(g^{-1})$
(say with a finite support)

assume that $\text{supp} \mu$ generate $G$.

$\mu$ defines a random walk on $G$:

start at $e$, \( g \overset{\mu(h)}{\longrightarrow} gh \) - transitions

Right convolution with $\mu$ defines a self-adjoint Markov operator $R_{\mu} : l^2(G) \to l^2(G)$

\[
(R_{\mu} f)(x) = \sum_{g \in G} f(xg) \mu(g)
\]
Harmonic functions and Poisson–Furstenberg boundary.

\[ M = \mathbb{R}^\mu - \text{Markov operator} \quad \Delta = \text{id} - M - \text{Laplace operator} \]

A function \( f \) on \( G \) is said to be harmonic iff \( Mf = f \) (or \( \Delta f = 0 \)).

There is a \( G \)-space \( (X, \nu) \) with \( \nu \) being \( \mu \)-stationary (\( \mu + \nu = \nu \)) s.t. \( \nu \) bounded harmonic function \( f(x) \) on \( G \) has a (unique) presentation

\[ f(y) = \int_X \varphi(y, x) \, d\nu(x) \]

\( \varphi \in L^\infty(X, \nu) \)
Poisson boundary is trivial \iff \forall \text{ bounded harmonic function is constant} \quad \text{(Liouville Property)}

Th. [Kaimanovich and Vershik 82]

1) The Poisson boundary of $\mathbb{F}_2 \mathbb{Z}^d$, $d=1, 2$, $|F_1|<\infty$ is trivial.

2) The Poisson boundary of $\mathbb{F} \mathbb{Z}^d$, $d \geq 3$, $|F_1|<\infty$ is nontrivial.

3) [A. Erschler] When $d \geq 5$ the Poisson boundary is equal to the space of limit configurations.
The return probabilities

\[ p(t) = p_{\mu,\mu}^{(e)} = \mu^{(e)}\{\{e\}\} = \langle R_{\mu}^t(\delta_e), \delta_e \rangle_{p^2(c)} \]

\( \delta_e \) - Delta function

Laplace operator \( \Delta = \text{id} - R_{\mu} \)

The \( L^2 \)-isometric profile \( \Lambda : [1, +\infty) \rightarrow (0, \infty) \)

\[ \Lambda(x) = \inf_{1 \leq |\Omega| \leq x} \lambda_1(\Omega), \quad \lambda_1(\Omega) = \inf_{\text{supp} f \subset \Omega} \frac{\langle \Delta(f), f \rangle}{\|f\|_2^2} \]

\( \Lambda \) is a decreasing right-continuous step function
$\Delta \in R \in N(G)$ - von Neumann algebra generated by right regular representation

$\text{tr}_G : N(G) \rightarrow \mathbb{C}$ - von Neumann trace:

$\text{tr}_G (A) = \langle A(\delta_e), \delta_e \rangle_{l^2(G)}$

For a self-adjoint operator $A \in N(G)$ the spectral projection

$E^A_\chi = \chi (-\infty, \chi) (A) \in N(G)$.

The spectral distribution of $\Delta$ is $N : [0, \infty) \rightarrow [0, \infty)$

$N(\lambda) = \text{tr}_G (E^A_\lambda)$ (spectral measure)
The group $G$ is non-amenable if and only if one (then all) of the following assertions holds:

1. Near infinity, $p_\mu(2t) \sim e^{-t}$ (H. Kesten)
2. Near infinity, $\Lambda_\mu(x) \sim 1$ ($\exists c > 0$ s.t. $\Lambda_\mu(x) \geq c$ for all $x$)
3. There exists $0 < \lambda_0$ s.t. $N_\mu(\lambda) = 0$ for all $0 \leq \lambda < \lambda_0$
| $\mathbb{F} \mathbb{Z} G$, $|\mathbb{F}| < \infty$ | $p(z) \rightarrow \infty$ | $N(\lambda)$ as $\lambda \rightarrow 0$ | $\wedge(x)$ as $x \rightarrow \infty$ |
|---|---|---|---|
| $G$ is polyn. of degree $d$ | $e^{-\frac{t}{d+2}}$ | $e^{-\lambda^{-\frac{1}{d}}}$ | $(-\log x)^{-\frac{1}{d}}$ |
| $H \mathbb{Z} G$, $|H| = \infty$ | $e^{-\lambda^{-\frac{1}{d+2}}(\log(1))^{-\frac{2}{d+2}}}$ | $e^{-\lambda^{1/2} \log(1)}$ | $\left(\frac{\log x}{\log \log x}\right)^{-\frac{1}{d}}$ |
| $H$ is polyn. gr. of $H$ | $e$ | $e$ | $e$ |
| $G$ is polyn. gr. of $G$ | $e^{\frac{1}{d+2}(\log(1))^{-\frac{2}{d+2}}}$ | $e^{-\exp_{k-1}(\frac{1}{2} \log(1))}$ | $\left(\frac{\log x}{\log \log x}\right)^{-\frac{k-1}{d}}$ |
| $\mathbb{Z} \mathbb{Z}(\cdots (\mathbb{Z} \mathbb{Z}(\mathbb{Z} \mathbb{Z})\cdots)$ | $\exp_{k-1}(\frac{1}{2} \log(1))$ | $e$ | $e$ |
| $k$-times iterated wreath product, $k \geq 2$ | $e$ | $\left(\frac{\log x}{\log \log x}\right)^{-\frac{k-1}{d}}$ | $\left(\frac{\log x}{\log \log x}\right)^{-\frac{k}{d}}$ |
| $\mathbb{F} \mathbb{Z}(\cdots (\mathbb{F} \mathbb{Z}(\mathbb{F} \mathbb{Z})\cdots)$ | $\frac{1}{(\log_{k-1}(1))^2}$ | $-\frac{t}{(\log_{k-1}(t))^2}$ | $-\frac{t}{(\log_{k-1}(t))^2}$ |
| $|\mathbb{F}| < \infty$, $k$ times iterated wreath product, $k \geq 2$ | $e$ | $-\exp_{k-1}(\frac{1}{2})$ | $\left(\frac{\log x}{\log \log x}\right)^{-\frac{k-1}{d}}$ |
SELF-SIMILARITY and actions on rooted trees
\[ S_p \cdots \rightarrow S_p \cong \text{Aut } T_{p,n} \text{ - p-regular tree of depth } n \]

\[ \mathbb{Z}_p \cdots \rightarrow \mathbb{Z}_p \cong \text{Aut } T_{p,n} \]  
Sylow p-subgroup
\[ S_p \ast S_p \ast \ldots \ast S_p \ast \ldots \cong \text{Aut } T_p \]
infinite iterated wreath product

\[ G \leq \text{Aut } T \]

\[ g = (g_0, \ldots, g_{p-1})^\sigma, \ \sigma \in S_p \]
sections

\[ G \text{ is self-similar if } \forall g, g_i \in G \text{ (after identification of } T_i \text{ with } T) \ i = 0, \ldots, p-1. \]

\[ \Leftrightarrow G = G(A), \ A \text{ - automaton of Mealy type.} \]
Example.

Zuk and Grigorchuk 2001

1) \( G(A) \cong \mathbb{Z} \) under the map \( a \rightarrow b^{-1}c, b \rightarrow b \).

2) The action of \( \mathbb{Z} \) given by automaton \( \mathcal{A} \) on the boundary \( \partial T \) of binary tree is essentially free [in contrast to branch actions which are extremely non-free, like for \( G = \langle a, b, c \rangle \)].

3) The subgroup \( \text{St}_L(1) \) of index 2 in \( \mathbb{Z} \) is isomorphic to \( \mathbb{Z} \) (a counterexample to Benjamini conjecture raised 5 years later).

\( \Rightarrow \) \( \mathbb{Z} \) is scale invariant (later was developed by V. Nekrashevych and C. Pete).
The Markov operator $M = \lambda \left( \frac{a + a^{-1} + b + b^{-1}}{4} \right)$ has a pure point spectrum: the eigenvalues are $\cos \frac{p}{q} \pi$, $q \in \mathbb{Z}$, $(p, q) = 1$, $0 \leq p \leq q$.

and

$$\dim_{\text{VN}} \ker (M - \cos \frac{p}{q} \pi I) = \frac{1}{2q - 1} = \text{mass of the spectral measure at point } \cos \frac{p}{q}$$

(This was used later to give a counter-example to the strong Atiyah Conjecture, discussed later.)

Gerry 2011 The essential freeness of the action of some self-similar groups on $\mathbb{Q}$ and the recurrent trace were used to construct asymptotic expanders.
The Atiyah Problem and the Lamplighter.

M. Atiyah 1976

\( (M, g) \) - closed Riemannian manifold

\( \tilde{M} \) - universal covering

\( \beta^p_{(2)} (M, g) \) - \( L^2 \)-Betti number (measure the size of the space of harmonic square-integrable \( p \)-forms on \( \tilde{M} \))

A priori, \( \beta^p_{(2)} (M) \in \mathbb{R} \) but

\[ \beta^p_{(2)} (M) = \beta^p_{(2)} (\tilde{M}, \pi_1 (M)) \]

Euler characteristic

\[ \chi (M) = \sum_{p=0}^{\infty} (-1)^p \beta^p_{(2)} (M) \]
The Atiyah Problem. "A priori, the numbers $\beta^p_{(n)}(M)$ are real. Give examples where they are not integral and even perhaps irrational."

The problem was converted into a bunch of Conjectures for manifolds and for groups.

$G$ - countable group, $\mathcal{N}(G)$ - von Neumann algebra
$\text{tr}_V -$ von Neumann trace, $\text{tr}_V(T) = \langle T\delta_e, \delta_e \rangle$
$C[G], \mathbb{R}[G], \mathbb{Q}[G]$ - group rings over $\mathbb{C}, \mathbb{R}, \mathbb{Q}$

$A \in C[G]$ - symmetric element $\rightarrow \lambda_A$ - right convolution operator (self-adjoint oper.)
We are interested in
\[ \dim_{\mathcal{V}_N} \ker \lambda_A = \text{tr}_{\mathcal{V}_N} (P_A) \]

Problem (The Atiyah problem for a group $G$): What is the set of \( \dim \ker \lambda_A \) when \( A \in \mathbb{Q}[G] \) (or \( \mathbb{Z}[G] \))?

If $G$ is finitely presented and $\theta = \dim (\ker \lambda_A)$, for $A \in \mathbb{Z}[G]$ then there exists a closed manifold $M$, with $\pi_1 (M) \cong G$ and such that one of the $L^2$-Betti numbers of $M$ is equal to $\theta$. 
If $G$ is recursively presented then $G$ embeds into a finitely presented group $H$ and $A \in \mathbb{Z}[G]$ becomes the element of $\mathbb{Z}[H]$ and $\dim_G(\ker \lambda_A) = \dim_H(\ker \lambda_A)$.

$$\omega(G) = \{ \dim(\ker \lambda_A) : A \in \text{QLG} \} = \ell^2\text{-complexity of } G$$

A. Zuk and G. Rii 2001

Using the realization of the lamplighter $\mathbb{L} = \mathbb{Z} \ast \mathbb{Z}$ as automaton group $\Gamma$ fixes the action of $\mathbb{L}$ on the boundary $\partial T$ of binary tree and (implicitly) the recurrent trace $\hat{\mathbb{L}}$ on $C_\partial$ ($C^*$-algebra generated by Koopman representation).
showed that $\frac{1}{3} \in C(X)$.

This was used by P. Linnell, T. Schick, A. Zuk and GRR to construct a 7-dimensional smooth oriented Riemannian manifold $(M, g)$ with $\theta^3_{(a)}(M) = \frac{1}{2}$ and $\pi_1(M) \cong \langle a, b, s \mid a^2 = [b, s] = [b^{-1}ab, a] = 1, s^{-1}as = [a, b] \rangle$ Baumslag–Remeslennikov group (ascending HNN-extension of $\mathbb{Z}$).

[A counter-example to a strong version of the Atiyah Conj.]

T. Austin 2009. For some left-invariant subspace $V \subset \oplus_{F_2} \mathbb{Z}_2$, the finitely generated group $(\oplus_{F_2} \mathbb{Z}_2) \rtimes F_2$ admits a group ring element with rational coefficients.
whose kernel has irrational (and even transcendental) von Neumann dimension.

What about recursively presented examples?

3. L. Grabowski  April 2010 1) The set of von Neumann dimensions arising from finitely generated groups is precisely the set of non-negative real numbers.

2) The set of von Neumann dimensions arising from finitely presented groups contains all numbers with recursive binary expansions.
3) Let $G = \langle a, b, s \rangle = \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z}$ be a Baumslag–Remeslennikov group and $S = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Then the following group gives rise to the irational von Neumann dimension

$$G \times G \times G \times (\hat{S} \rtimes \text{Aut}(S)),$$

where the semidirect product is taken with respect to the natural action of $\text{Aut}(S)$ on the Pontryagin dual $\hat{S}$.

4) M. Pichot, T. Schick, A. Zuk May 2010 showed that $C(G)$ contains irrational number for a group $G$ of type

$$\left( \bigoplus \mathbb{Z} / \sqrt{V} \bigg) \times \Gamma$$

where $V$ is a suitable $\Gamma$-invariant subspace of $\bigoplus \mathbb{Z}$ and $\Gamma$ is either $\mathbb{F}_2$ or $\mathbb{Z} \rtimes \mathbb{Z}$. 
contains an irrational algebraic number.

L. Grabowski, September 1 2010 C(\mathbb{Z}_m \times \mathbb{Z}) contains transcendental numbers. (m \geq 2)

Two questions of Grabowski.

Q1. What is C(\mathbb{Z}_2 \times \mathbb{Z})?

Q2. Is it the case that C(\mathbb{C}) \notin \mathbb{Q} is equivalent to \mathbb{Z}_m \times \mathbb{Z} \rightarrow \mathbb{C} for some m?

Problem. What is the spectral measure of Laplace operator on \mathbb{Z} for the standard system of generators a, b? Does it has a continuous component?