1. $V$ is an $n$-dimensional vector space over $\mathbb{R}$.

   a) Define a tensor of type $(r, s)$.

   First, we define a multilinear map to be a function $\sigma : V_1 \times \ldots \times V_r \to \mathbb{R}$ which is linear in each $V_i$:
   \[
   \sigma(v_1, \ldots, av_i + bu_i, \ldots, v_r) = a\sigma(v_1, \ldots, v_i, \ldots, v_r) + b\sigma(v_1, \ldots, u_i, \ldots, v_r)
   \]

   Then for $r, s \in \mathbb{N}$, $\sigma$ is a tensor of type $(r, s)$ iff it is a multilinear map $\sigma : V \times \ldots \times V^* \times \ldots \times V^* \to \mathbb{R}$.

   $r$ is called the covariant order, and $s$ the contravariant order, of $\sigma$.

   b) Explain how $\mathcal{T}^r_s(V)$ forms a vector space of dimension $n^{r+s}$.

   c) i) For $\mathcal{T}^*(V) = \bigoplus_{k=0}^{\infty} \mathcal{T}^k(V)$, explain how $(\mathcal{T}^*(V), +, \otimes)$ forms an associative algebra.

   See [Booth] p.207, Thm 6.2

   ii) Is the tensor product $\otimes$ commutative?

   No, it is not commutative. Let $V = \mathbb{R}^2$, and let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be vectors in $V$. Define $\varphi : V \times V \to \mathbb{R}$ and $\psi : V \times V \to \mathbb{R}$ by
   \[
   \varphi(u, v) = \langle u, v \rangle = u_1v_1 + u_2v_2
   \]

   $\psi(u, v) = \det(u, v) = u_1v_2 - u_2v_1$

   For an example, we use four randomly selected vectors of $\mathbb{R}^2$:

   $s = (0, 1), t = (1, 1), u = (1, 3), v = (1, 0)$

   and compute
   \[
   \varphi \otimes \psi(s, t, u, v) = \varphi \otimes \psi((0, 1), (1, 1), (1, 3), (1, 0))
   = \varphi((0, 1), (1, 1)) \psi((1, 3), (1, 0))
   = (0 + 1)(0 - 3)
   = -3
   \]

   and
   \[
   \psi \otimes \varphi(s, t, u, v) = \psi \otimes \varphi((0, 1), (1, 1), (1, 3), (1, 0))
   = \psi((0, 1), (1, 1)) \varphi((1, 3), (1, 0))
   = (0 - 1)(1 + 0)
   = -1
   \]
2. Explain the tensor fields $\mathcal{T}_r^s(M), \mathcal{T}^r(M), \mathcal{T}_s(M)$ over a manifold $M$. How is a tensor field written in local coordinates?

See [Booth] p.209

3. a) Define the alternating and symmetric tensors.

Let $V$ be a vector space, and let $\varphi \in \mathcal{T}^r(V)$ be some tensor. We say $\varphi$ is symmetric iff $\forall i, j = 1, 2, \ldots, r$, we have

$$\varphi (v_1, \ldots, v_i, \ldots, v_j, \ldots, v_r) = \varphi (v_1, \ldots, v_j, \ldots, v_i, \ldots, v_r)$$

We say $\varphi$ is alternating iff $\forall i, j = 1, 2, \ldots, r$, we have

$$\varphi (v_1, \ldots, v_i, \ldots, v_j, \ldots, v_r) = -\varphi (v_1, \ldots, v_j, \ldots, v_i, \ldots, v_r)$$

b) Define the alternating and symmetrizing operators.

The action of the alternating operator $A : T^r(V) \to T^r(V)$ on a tensor $\varphi$ is defined pointwise as

$$(A\varphi) (v_1, \ldots, v_r) = \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} (\text{sgn} \sigma) \varphi (v_{\sigma(1)}, \ldots, v_{\sigma(r)})$$

The action of the symmetrizing operator $S : T^r(V) \to T^r(V)$ on a tensor $\varphi$ is defined pointwise as

$$(S\varphi) (v_1, \ldots, v_r) = \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} \varphi (v_{\sigma(1)}, \ldots, v_{\sigma(r)})$$

$\mathfrak{S}_r$ is the symmetric group of all permutations on $r$ letters.

c) Define the wedge product: $\wedge : \bigwedge^k(V) \times \bigwedge^l(V) \to \bigwedge^{k+l}(V)$.

$$(\varphi, \psi) \overset{\wedge}{\longrightarrow} (\varphi \wedge \psi) \overset{\text{def}}{=} \frac{1}{(r+s)!} A (\varphi \otimes \psi)$$

d) Give an example $\omega \in \bigwedge^k, \eta \in \bigwedge^l$ such that $\omega \otimes \eta \notin \bigwedge^{k+l}$.

Consider the tensors $dx, dy \in \bigwedge^1$, considered as alternating tensors on $V$, and observe how they act on the vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$:

$$dx \otimes dy ((u_1, u_2), (v_1, v_2)) = dx (u_1, u_2) \cdot dy (v_1, v_2) = u_1 v_2$$

but

$$dx \otimes dy ((v_1, v_2), (u_1, u_2)) = dx (v_1, v_2) \cdot dy (u_1, u_2) = v_1 u_2.$$
4. a) If dim$_{\mathbb{R}} V = n$, then what is the dimension of $\bigwedge^k(V)$? What is its basis?

If the vector space $V$ has basis $\{v_1, \ldots, v_n\}$, then the basis of $\bigwedge^k(V)$ is

$$B = \{v_{i_1} \wedge \ldots \wedge v_{i_k} : i_1 < \ldots < i_k\}.$$ 

Every element of $\bigwedge^k(V)$ can be written in the form $\sum_{i=1}^n a_i v_i$, so using the rule $x \wedge y = -y \wedge x$, it is clear that $v_1 \wedge \ldots \wedge v_n$ generates $\bigwedge^k(V)$. Since the $v_i$ are linearly independent, $B$ forms a basis. The final thing to note is that the rule $x \wedge y = -y \wedge x$ forces any basis element to have distinct factors, and no more than $k$. This is because if $v_i = v_j$, then $v_1 \wedge \ldots \wedge v_i \wedge v_j \wedge v_k = 0$. See the next problem for more details.

Based on this, the cardinality of the basis can be calculated as simple counting argument. Since any basis element looks like $v_{i_1} \wedge \ldots \wedge v_{i_k}$, there are $k$ components to choose and $n$ possibilities to choose from. Order does not matter, because $x \wedge y = -y \wedge x$, so as generators of a vector space, $x \wedge y$ and $y \wedge x$ are essentially equivalent. Hence, the number of elements in any basis, and thus the dimension of $\bigwedge^k(V)$ is $\binom{n}{k}$.

b) Prove that $\bigwedge^k(V) = \{0\}$ if $k > n$.

Since the basis of $\bigwedge^k(V)$ is $B = \{v_{i_1} \wedge \ldots \wedge v_{i_k} : i_1 < \ldots < i_k\}$ and $k > n$, every basis element looks like:

$$v_{i_1} \wedge \ldots \wedge v_{i_n} \wedge v_{i_{n+1}} \ldots \wedge v_{i_k}$$

Consider the possibilities for $v_{i_{n+1}}$. $v_{i_{n+1}}$ must be one of the $\{v_i\}_{i=1}^n$ that form a basis of $V$. Suppose $v_{i_{n+1}} = v_j$. But $v_j$ is already one of the initial $v_{i_1} \wedge \ldots \wedge v_{i_n}$, so we are in the case where $v_j$ occurs twice, which implies immediately that the entire basis element $v_{i_1} \wedge \ldots \wedge v_{i_n} \wedge v_{i_{n+1}} \ldots \wedge v_{i_k}$ is $0$. To see this, first note that we can reorder the basis element so that the two $v_j$s are adjacent, without changing the value of the element. Then by swapping positions of the two $v_j$, we get

$$v_{i_1} \wedge \ldots \wedge v_j \wedge v_j \ldots \wedge v_{i_k} = -v_{i_1} \wedge \ldots \wedge v_j \wedge v_j \ldots \wedge v_{i_k}$$

which shows that both sides are $0$. Since every basis element is $0$, the entire basis is $0$, and thus $\bigwedge^k(V) = \{0\}$.

c) For $\bigwedge^*(V) = \bigotimes_{k=0}^\infty \bigwedge^k(V)$, what is dim $\bigwedge^*(V)$? Explain how $(\bigwedge(V), +, \wedge)$ forms an associative algebra. Is it commutative?

We compute the dimension of using a technique from algebra:

$$[\bigwedge^*(V) : \mathbb{R}] = [\bigwedge^*(V) : C^\infty(V)] \cdot [C^\infty(V) : \mathbb{R}] = n \cdot \infty = \infty$$
d) Is \((\wedge(V), +, \land)\) a subalgebra of \((\mathcal{T}^*(V), +, \otimes)\)?

Yes. Clearly \((\wedge(V), +, \land) \subset (\mathcal{T}^*(V), +, \otimes)\), and we have shown above that \(\wedge(V)\) actually does form an algebra under these operations. The key fact that distinguishes this case from 3(d) is that for tensors \(\varphi\) and \(\psi\), \(\varphi \otimes \psi\) may not be alternating, but the wedge product is defined in terms of the alternating operator, so that \(\varphi \land \psi\) is alternating by construction. This is why \(\wedge(V)\) is closed under \(\land\).

5. a) Explain a \(k\)-form \(\omega \in \bigwedge^k(M)\) on a manifold.

b) Explain the exterior/Grassman algebra on a manifold.

c) Let \(F : M \to N\) be a \(C^\infty\) map between two manifolds \(M, N\). Describe the pullback map \(F^* : \bigwedge(N) \to \bigwedge(M)\).

d) Prove that \(F^*\) is an algebra homomorphism.

\[
F^*(\omega + \eta) = (\omega + \eta) \circ F = \omega \circ F + \eta \circ F = F^*\omega + F^*\eta
\]

Now we need to show \(F^*(\omega \land \eta) = F^*\omega \land F^*\eta\), so we begin by showing \(F^*(\omega \otimes \eta) = F^*\omega \otimes F^*\eta\) as follows:

\[
F^*(\omega \otimes \eta)(u, v) = ((\omega \otimes \eta) \circ F)(u, v)
= (\omega \otimes \eta)(F(u), F(v))
= \omega \circ F(u) \cdot \eta \circ F(v)
= F^*\omega(u) \cdot F^*\eta(v)
= (F^*\omega \otimes F^*\eta)(u, v)
\]

Now we proceed with \(\land\):

\[
F^*(\omega \land \eta)(u, v)
= (\omega \land \eta)(F(u), F(v))
= \binom{r+s}{r,s,t!}\mathcal{A}(\omega \otimes \eta)(F(u), F(v))
= \binom{r+s}{r,s,t!}\sum_{\sigma \in \mathcal{S}_{k+j}} (\text{sgn } \sigma)(\omega \otimes \eta)(F(u), F(v))
= \binom{r+s}{r,s,t!}\sum_{\sigma \in \mathcal{S}_{k+j}} (\text{sgn } \sigma)(F^*\omega \otimes F^*\eta)(u, v)
= \binom{r+s}{r,s,t!}\mathcal{A}(F^*\omega \otimes F^*\eta)(u, v)
= \binom{r+s}{r,s,t!}\mathcal{A}(F^*\omega \land F^*\eta)(u, v)
\]

Thus, \(F^*(\omega \land \eta) = F^*\omega \land F^*\eta\). □
6. a) Define the exterior differentiation operator $d : \bigwedge^k(M) \to \bigwedge^{k+1}(M)$.

$d$ is the unique $\mathbb{R}$-linear map $d : \bigwedge^*(M) \to \bigwedge^*(M)$ satisfying

i) for $f \in C^\infty(M) = \bigwedge^0(M)$, we have $d(f) = df$, the differential of $f$.

ii) $d(\theta \wedge \sigma) = d\theta \wedge \sigma + (-1)^r \theta \wedge d\sigma, \forall \theta \in \bigwedge^r, \forall \sigma \in \bigwedge^k$

iii) $d^2 = 0$

b) Prove that $d^2 = 0$.

For $\omega \in \bigwedge^p(M)$, we can write $\omega = \sum \omega_{i_1...i_p} dx_{i_1} \wedge ... \wedge dx_{i_p}$ in local coordinates. Since $d$ is $\mathbb{R}$-linear, it will suffice to consider the case $d\left(d\left(\omega_{i_1...i_p} dx_{i_1} \wedge ... \wedge dx_{i_p}\right)\right)$. Then

\[
d^2 \omega_i = d\left(d\left(\omega_{i_1...i_p} dx_{i_1} \wedge ... \wedge dx_{i_p}\right)\right)
\]
\[
= d\left[\sum_{k=1}^n \left(\frac{\partial \omega_{i_1...i_p}}{\partial x_k} dx_k\right) \wedge dx_{i_1} \wedge ... \wedge dx_{i_p}\right]
\]
\[
= d\left[\sum_{k=1}^n \frac{\partial \omega_{i_1...i_p}}{\partial x_k} dx_k \wedge dx_{i_1} \wedge ... \wedge dx_{i_p}\right]
\]
\[
= \sum_{k=1}^n \sum_{j=1}^n \left[\frac{\partial^2 \omega_{i_1...i_p}}{\partial x_k \partial x_j} dx_j \wedge dx_k \wedge dx_{i_1} \wedge ... \wedge dx_{i_p}\right]
\]
\[
= \sum_{1 \leq k < j \leq n} \left[\frac{\partial^2 \omega_{i_1...i_p}}{\partial x_k \partial x_j} dx_j \wedge dx_k \wedge dx_{i_1} \wedge ... \wedge dx_{i_p}\right].
\]

but then $\frac{\partial^2 \omega_{i_1...i_p}}{\partial x_k \partial x_j} = \frac{\partial^2 \omega_{i_1...i_p}}{\partial x_k \partial x_j}$, by the equality of mixed partial derivatives. So each term of the sum, and hence the entire sum, is 0.

c) Define the closed and exact differential forms.

$\omega \in \bigwedge^p(M)$ is a closed form iff $d\omega = 0$, i.e., $\omega \in \text{Ker } d$.

$\omega \in \bigwedge^p(M)$ is an exact form iff $\exists \eta \in \bigwedge^{p-1}(M)$ such that $d\eta = \omega$, i.e., $\omega \in \text{Im } d$.

\[
\bigwedge^{p-1}(M) \xrightarrow{d} \bigwedge^p(M) \xrightarrow{d} \bigwedge^{p+1}(M)
\]

7. a) Let $\omega = \sum_{i=1}^n \omega_i dx_i$ be a 1-form in $\mathbb{R}^n$. Show $\omega$ is closed $\iff \frac{\partial \omega_i}{\partial x_j} = \frac{\partial \omega_j}{\partial x_i}, \forall i, j$.

$\implies$ Assume $\omega$ is closed so that $d\omega = 0$. Then

\[
0 = d\omega = \sum_{k=1}^n \left(\sum_{i=1}^n \frac{\partial \omega_i}{\partial x_k} dx_k\right) \wedge dx_i
\]
\[
= \sum_{k=1}^n \left(\sum_{i=1}^n \frac{\partial \omega_i}{\partial x_k} dx_k\right) \wedge dx_i
\]
\[
= \sum_{1 \leq k < i \leq n} \left(\frac{\partial \omega_i}{\partial x_k} - \frac{\partial \omega_k}{\partial x_i}\right) dx_k \wedge dx_i
\]
But the basis elements \( \{dx_k \wedge dx_i\} \) are linearly independent, so this implies
\[
\frac{\partial \omega_i}{\partial x_k} - \frac{\partial \omega_k}{\partial x_i} = 0 \quad \forall i, k \quad \implies \quad \frac{\partial \omega_i}{\partial x_k} = \frac{\partial \omega_k}{\partial x_i} \quad \forall i, k.
\]
\[
\iff \quad \frac{\partial \omega_i}{\partial x_k} = \frac{\partial \omega_k}{\partial x_i} = 0, \text{ which immediately gives}
\]
\[
d\omega = \sum_{1 \leq k < i \leq n} \left( \frac{\partial \omega_i}{\partial x_k} - \frac{\partial \omega_k}{\partial x_i} \right) dx_k \wedge dx_i = 0.
\]
\[
\square
\]

b) Find necessary and sufficient conditions for a 2-form \( \omega = Cdx \wedge dy + Ady \wedge dz + Bdz \wedge dx \) to be closed in \( \mathbb{R}^3 \), where \( A, B, C \) are functions in \( \mathbb{R}^3 \).

We put \( \omega_{12} = C, \omega_{23} = A, \omega_{13} = -B \), and consider \( \omega = \sum_{1 \leq i < j \leq 3} \omega_{ij} dx_i \wedge dx_j \).

Now
\[
d\omega = \left( \frac{\partial C}{\partial x} dx \right) \wedge dx \wedge dy + \left( \frac{\partial B}{\partial y} dy \right) \wedge dy \wedge dz + \left( \frac{\partial B}{\partial z} dz \right) \wedge dz \wedge dx
\]
\[
+ \left( \frac{\partial A}{\partial x} dx \right) \wedge dy \wedge dz + \left( \frac{\partial A}{\partial y} dy \right) \wedge dx \wedge dy + \left( \frac{\partial A}{\partial z} dz \right) \wedge dx \wedge dy
\]
\[
+ \left( \frac{\partial B}{\partial x} dx \right) \wedge dx \wedge dz + \left( \frac{\partial B}{\partial y} dy \right) \wedge dx \wedge dz + \left( \frac{\partial B}{\partial z} dz \right) \wedge dx \wedge dz
\]
\[
+ d(dx \wedge dy)
\]
\[
+ d(dy \wedge dz)
\]
\[
+ d(dz \wedge dx)
\]

Now by the basic properties of the alternating product,
\[
dx \wedge dy = -dy \wedge dx \quad \implies \quad dx \wedge dx = -dx \wedge dx \quad \implies \quad dx \wedge dx = 0
\]

Using the rule that \( dx \wedge dx = 0 \), we see that only the third, fourth, and eighth terms of the above sum are nonzero, i.e.,
\[
d\omega = \left( \frac{\partial C}{\partial z} dz \right) \wedge dx \wedge dy + \left( \frac{\partial A}{\partial x} dx \right) \wedge dy \wedge dz + \left( \frac{\partial B}{\partial y} dy \right) \wedge dx \wedge dz
\]
\[
= \frac{\partial A}{\partial x} dx \wedge dy \wedge dz + \frac{\partial B}{\partial y} dy \wedge dx \wedge dy + \frac{\partial C}{\partial z} dz \wedge dx \wedge dy
\]
\[
= \left( \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) dx \wedge dy \wedge dz
\]

Thus we obtain \( d\omega = 0 \iff \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0 \).
\[
\square
\]

c) Show that a 2-form \( \omega = \sum_{i<j} a_{ij} dx_i \wedge dx_j \) in \( \mathbb{R}^n \) is closed \( \iff \frac{\partial a_{ij}}{\partial x_k} - \frac{\partial a_{ik}}{\partial x_j} + \frac{\partial a_{jk}}{\partial x_i} = 0 \ \forall i, j, k \)

\[
d\omega = \sum_{1 \leq i < j \leq n} \sum_{k=1}^n \frac{\partial \omega_{ij}}{\partial x_k} dx_k \wedge dx_i \wedge dx_j
\]
8. Consider the 1-form \( \omega = \frac{xdy - ydx}{x^2 + y^2} \) in \( \mathbb{R}^2 \setminus \{(0,0)\} \). Is \( \omega \) closed? Is \( \omega \) exact?

\[
\omega = \sum_{i=1}^{2} \omega_i dx_i = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy
\]

So applying \( d \) gives

\[
d\omega = \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\partial \omega_i}{\partial x_k} dx_i
\]

Which shows that \( \omega \) will be closed precisely when

\[
\frac{\partial}{\partial y} \left( \frac{-y}{x^2 + y^2} \right) = \frac{\partial}{\partial x} \left( \frac{x}{x^2 + y^2} \right).
\]

Now

\[
\frac{\partial}{\partial y} \left( \frac{-y}{x^2 + y^2} \right) = \frac{(x^2 + y^2) + y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}
\]

and

\[
\frac{\partial}{\partial x} \left( \frac{x}{x^2 + y^2} \right) = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}
\]

shows that \( \omega \) is closed.

9. a) In \( \mathbb{R}^3 \), determine which of the following forms are closed and which are exact:

i) \( \varphi = yzdx + xzdy + xydz \)

Consider \( \varphi = \omega_1 dx_1 + \omega_2 dx_2 + \omega_3 dx_3 \). Then \( \varphi \) is closed because

\[
d\varphi = \sum_{1 \leq k < i \leq 3} \left[ \frac{\partial \omega_i}{\partial x_k} - \frac{\partial \omega_k}{\partial x_i} \right] dx_k \wedge dx_i
\]

\[
= \left( \frac{\partial \omega_2}{\partial x_1} - \frac{\partial \omega_1}{\partial x_2} \right) dx_1 \wedge dx_2
\]

\[
+ \left( \frac{\partial \omega_3}{\partial x_1} - \frac{\partial \omega_1}{\partial x_3} \right) dx_1 \wedge dx_3 + \left( \frac{\partial \omega_3}{\partial x_2} - \frac{\partial \omega_2}{\partial x_3} \right) dx_2 \wedge dx_3
\]

\[
= \left( \frac{\partial (xz)}{\partial x} - \frac{\partial (yz)}{\partial y} \right) dx \wedge dy
\]

\[
+ \left( \frac{\partial (xy)}{\partial x} - \frac{\partial (yz)}{\partial z} \right) dx \wedge dz + \left( \frac{\partial (xy)}{\partial y} - \frac{\partial (xz)}{\partial z} \right) dy \wedge dz
\]

\[
= (z - z) dx \wedge dy + (y - y) dx \wedge dz + (x - x) dy \wedge dz
\]

\[= 0\]

If we define \( \psi \in \wedge^0(\mathbb{R}^3) \) by \( \psi = xyz \), then \( \varphi \) is exact because

\[
d\psi = \frac{\partial}{\partial x} (xyz) dx + \frac{\partial}{\partial y} (xyz) dy + \frac{\partial}{\partial z} (xyz) dz
\]

\[= yzdx + xzdy + xydz\]

\[= \varphi\]

\[
\text{Note that } \varphi \text{ exact } \Rightarrow \varphi \text{ closed, so some of these calculations are unnecessary.} \]

\[\blacksquare\]
ii) \( \varphi = xdx + x^2y^2dy + xzdz \)

\( \varphi \) is not closed because

\[
d\varphi = \left( \frac{\partial (x^2y^2)}{\partial x} - \frac{\partial (x^2y^2)}{\partial y} \right) dx \wedge dy \\
+ \left( \frac{\partial (xz)}{\partial x} - \frac{\partial (x)}{\partial z} \right) dx \wedge dz + \left( \frac{\partial (xz)}{\partial y} - \frac{\partial (x^2y^2)}{\partial z} \right) dy \wedge dz
\]

\[= (2xy^2 - 0) \, dx \wedge dy + (z - 0) \, dx \wedge dz + (0 - 0) \, dy \wedge dz\]

\[= 2xy^2 \, dx \wedge dy + z \, dx \wedge dz \neq 0\]

Since not closed \( \implies \) not exact, we know that \( \varphi \) cannot be exact. \( \blacksquare \)

iii) \( \varphi = 2xy^2 dx \wedge dy + zdy \wedge dz \)

\( \varphi \) is closed because

\[
d\varphi = \sum_{1 \leq i_1 < i_2 \leq 3} \left[ \sum_{k=1}^{3} \frac{\partial \omega_{i_1 i_2}}{\partial x_k} dx_k \right] \wedge dx_{i_1} \wedge dx_{i_2}
\]

\[= \left( \frac{\partial (2xy^2)}{\partial x} dx \right) \wedge dx \wedge dy \\
+ \left( \frac{\partial (2xy^2)}{\partial y} dy \right) \wedge dx \wedge dy + \left( \frac{\partial (2xy^2)}{\partial z} dz \right) \wedge dx \wedge dy
\]

\[+ (\frac{\partial z}{\partial x} dx) \wedge dy \wedge dz + (\frac{\partial z}{\partial y} dy) \wedge dy \wedge dz + (\frac{\partial z}{\partial z} dz) \wedge dy \wedge dz
\]

\[= 2y^2 \, dx \wedge dy + 4xy dy \wedge dx \wedge dy + 0dz \wedge dx \wedge dy \]

\[+ 0dx \wedge dy \wedge dz + 0dy \wedge dy \wedge dz + 1dz \wedge dy \wedge dz
\]

\[= 2y^2 \, dx \wedge dy + 4xydy \wedge dx \wedge dy + 1dz \wedge dy \wedge dz
\]

\[= 0
\]

If we define \( \psi \in \wedge^1(\mathbb{R}^3) \) by \( \psi = cdx + x^2y^2dy + zydz \), where \( c \in \mathbb{R} \), then \( \varphi \) is exact because

\[
d\psi = \left( \frac{\partial c}{\partial x} dx \right) \wedge dx + \left( \frac{\partial c}{\partial y} dy \right) \wedge dx + \left( \frac{\partial c}{\partial z} dz \right) \wedge dx
\]

\[+ \left( \frac{\partial (x^2y^2)}{\partial x} dx \right) \wedge dy + \left( \frac{\partial (x^2y^2)}{\partial y} dy \right) \wedge dy + \left( \frac{\partial (x^2y^2)}{\partial z} dz \right) \wedge dy
\]

\[+ \left( \frac{\partial (xy)}{\partial x} dx \right) \wedge dz + \left( \frac{\partial (xy)}{\partial y} dy \right) \wedge dz + \left( \frac{\partial (xy)}{\partial z} dz \right) \wedge dz
\]

\[= 0dx \wedge dx + 0dy \wedge dx + 0dz \wedge dx
\]

\[+ 2xy^2 \, dx \wedge dy + 2x^2ydy \wedge dy + 0dz \wedge dy
\]

\[+ 0dx \wedge dz + zdy \wedge dz + ydz \wedge dz
\]

\[= 2xy^2 \, dx \wedge dy + zdy \wedge dz
\]

\[= \varphi \]
10. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ taking $(x, y, z) \mapsto (s, t)$ be defined by

$$f(x, y, z) = (xy, yz + 1).$$

a) Let $\varphi = stds + dt$ be a 1-form in $\mathbb{R}^2$. Compute $f^*(\varphi)$.

$$f^*(\varphi) = \varphi \circ f,$$

so $s = xy$ and $t = yz + 1$, so we find the other components of $\varphi$ in terms of $x, y, z$ as

$$ds = \frac{\partial(xy)}{\partial x} dx + \frac{\partial(xy)}{\partial y} dy + \frac{\partial(xy)}{\partial z} dz = ydx + xdy,$$

$$dt = \frac{\partial(yz+1)}{\partial z} dz = zdy + ydz.$$

Then

$$f^*(\varphi) = \frac{s}{(xy)} \left( \frac{t}{(yz+1)} \right) \left( \frac{ds}{(ydx+xdy)} + \frac{dt}{(zdy+ydz)} \right)$$

$$= x^2 y^2 z dy + xy^3 z dx + x^2 y^2 dx + y dz + z dy$$

$$= (xy^3 z + xy^2) dx + (x^2 y^2 z + z + x^2 y) dy + y dz.$$

\[\square\]

b) Let $\varphi = st (ds \wedge dt)$ be a 1-form in $\mathbb{R}^2$. Compute $f^*(\varphi)$.

Using $ds, dt$ as calculated above, we obtain

$$f^*(\varphi) = (xy) (yz+1) \left( (ydx+xdy) \wedge (zdy+ydz) \right)$$

$$= (xy^2 z + xy) \left( yz dx \wedge dy + y^2 dx \wedge dz + xzd y \wedge dx + ydy \wedge dz \right)$$

$$= (xy^3 z^2 + xy^2 z) dx \wedge dy + (xy^4 z + xy^3) dx \wedge dz + (x^2 y^3 z + x^2 y^2)$$

\[\square\]

11. Let $\omega$ be a 1-form. For vector fields $X, Y$, prove the formula

$$d\omega \left( X, Y \right) = X \left( \omega(Y) \right) - Y \left( \omega(X) \right) - \omega \left( [X, Y] \right).$$

$X, Y \in \mathfrak{X}(M)$ and $\omega \in \bigwedge^1(M)$, so let $\omega = f dg$ where $f, g \in C^\infty$. It will suffice to prove that the formula is true locally, i.e., in a coordinate neighbourhood of each point. In each such neighbourhood, with coordinates $x_1, \ldots, x_n$, we have $\omega = \sum_{i=1}^n a_i dx_i$, by the definition of $\omega$ as a 1-form. Now the left side of the formula becomes

$$d\omega \left( X, Y \right) = df \wedge dg \left( X, Y \right)$$

$$= df(X) dg(Y) - dg(X) df(Y)$$

$$= (X f)(Y g) - (X g)(Y f)$$
and the right side of the formula becomes

\[
X \omega(Y) - Y \omega(X) - \omega([X,Y]) = X(f dg(Y)) - Y(f dg(X)) - f dg([X,Y]) \\
= X(f(Yg)) - Y(f(Xg)) - f(XYg - YXg) \\
= (Xf)(Yg) - (Xg)(Yf)
\]

12. a) Define a Riemannian metric.

A Riemannian metric on a differentiable manifold \( M \) is not actually a metric at all. Instead, it is the (rather misleading) name given to any positive definite covariant symmetric tensor of type \((2,0)\). More formulaically,

\[
ds^2 = \sum_{1 \leq i < j \leq n} g_{ij} dx_i \otimes dx_j,
\]

where \( g_{ij} = g_{ji}, \forall i, j \)

b) Describe two ways to construct a Riemannian metric on a manifold \( M \).

i) Using the Whitney Imbedding Theorem, we can imbed \( M \) into some \( \mathbb{R}^n \). Let \( dS^2_E \) denote the standard Euclidean metric of \( \mathbb{R}^n \). If we restrict \( dS^2_E \) to \( M \), we get a Riemannian metric on \( M \).

ii) First, we choose an open cover \( \{ A_\alpha \} \) of \( M \) and use Boothby V.4.1 to produce a regular covering \( \{ U_i, V_i, \varphi_i \} \) and Boothby V.4.4 to produce a \( C^\infty \) partition of unity \( \{ f_i \} \) subordinate to this cover. In a given coordinate neighbourhood \( U_i \), we define a “local” Riemannian metric by

\[
\Phi_i = \varphi_i^*(\psi),
\]

where \( \psi = dx_1^2 + dx_2^2 + \ldots + dx_n^2 \) is the Euclidean metric. Finally, define

\[
ds^2 = \Phi = \sum_i f_i \Phi_i
\]
to obtain a globally-defined Riemannian metric.

c) How does one make a metric out of the Riemannian metric?

Let \( \Phi \) be a a Riemannian metric defined on \( M \). By simply denoting \( \langle x, y \rangle = \Phi(x,y) \), the Riemannian metric gives an inner product to the tangent space \( T_p(M), \forall p \in M \). Let \( \gamma(t) \) be a \( C^1 \) curve on \( M \), and define \( p_0 = \gamma(0) \) and \( p_1 = \gamma(1) \). Define the length \( L \) of this curve, from \( p_0 \) to \( p_1 \), by

\[
L = \int_0^1 \left[ \Phi \left( \frac{dx}{dt}, \frac{dx}{dt} \right) \right]^{1/2} dt = \int_0^1 \sqrt{\left\langle \frac{dx}{dt}, \frac{dx}{dt} \right\rangle} dt = \int_0^1 \| \gamma'(t) \| dt
\]

We now obtain a metric \( d(x, y) \) by defining the distance from \( p_0 \) to \( p_1 \) as

\[
d(p_0, p_1) = \inf_{\gamma \in C^1} \left\{ \int_0^1 \| \gamma'(t) \| dt \right\}
\]
13. a) Define a volume element.

b) Compute the volume (with the induces metric of $\mathbb{R}^2$) in terms of the coordinates given by
   
   i) Stereographic projection.
   
   ii) Spherical coordinates (with $\rho = 1$).