A TUBE FORMULA FOR THE KOCH SNOWFLAKE CURVE, WITH APPLICATIONS TO COMPLEX DIMENSIONS.

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Current (i.e., unfinished) draught of the full version is available at http://math.ucr.edu/~epear se/koch.pdf.
Figure 1. The Koch curve $K$ (left) and the Koch snowflake $\Omega$ (right).
Goal: derive a formula for the $\varepsilon$-neighbourhood of the Koch curve (and snowflake).

We want to find a formula for

$$V(\varepsilon) = \text{area of shaded region} = \text{vol}_2\{x \in \Omega : d(x, \partial \Omega) < \varepsilon\}$$
Q: What use is $V(\varepsilon)$?
A: A precise formula for $V(\varepsilon)$ will help towards extending the theory of fractal strings into higher dimensions.

A fractal string is any bounded open subset of $\mathbb{R}$

$$\mathcal{L} := \{l_j\}_{j=1}^{\infty}, \quad \text{with } \sum_{j=1}^{\infty} l_j < \infty.$$ 

$$l_1 \geq l_2 \geq l_3 \geq \ldots,$$

or distinctly (with multiplicity):

$$l_1 > l_2 > l_3 > \ldots.$$

Idea/origin: comes from studying fractal subsets $\partial \mathcal{L} \subseteq \mathbb{R}$.

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**Figure 3. The Cantor Set**

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**Figure 4. The Cantor String**

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The Cantor String example has lengths
\[ \left\{ 3^{-(n+1)} \right\} \]
with multiplicities
\[ w_{3^{-(n+1)}} = 2^n. \]

\[ \begin{array}{cccccccc}
 & & & & l_1 & & & \\
 & & & l_2 & & & & \\
 & & l_3 & & & & & \\
 & l_4 & & l_5 & & l_6 & & l_7 \\
& & & & & & &
\end{array} \]

\[ \mathcal{CS} = \left\{ \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \ldots \right\} \]

The geometric zeta function of a string
\[ \zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s = \sum_{l} w_l l^s \]
encodes all this information.

Example:
\[ \zeta_{\mathcal{CS}}(s) = \sum_{n=0}^{\infty} 2^n 3^{-(n+1)s} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}. \]
Three key things about $\zeta_L$:
(1) Relates to the dimension of $\partial L$.
(2) Connects spectral and geometric properties.
(3) Gives an explicit formula for $V(\varepsilon)$.

For (1), recover the Minkowski dimension

$$D_{\partial L} := \inf\{t \geq 0 : V(\varepsilon) = O(\varepsilon^{1-t}) \text{ as } \varepsilon \to 0^+\} = \inf\{\sigma \geq 0 : \zeta_L(s) < \infty\}$$

Generalize and define the complex dimensions:

$$\mathcal{D} = \{\omega \in \mathbb{C} : \zeta_L \text{ has a pole at } \omega\}$$

**Theorem 1** (Structure of Complex Dimensions).

*If $\partial L$ is self-similar, then either*

(1) $\mathcal{D}$ is “periodic” and $\partial L$ is not measurable, or

(2) $\mathcal{D}$ is “quasiperiodic” and $\partial L$ is measurable.

Here, $\partial L$ is *(Minkowski) measurable* iff

$$\mathcal{M} = \mathcal{M}(D; \partial L) = \lim_{\varepsilon \to 0^+} V(\varepsilon)\varepsilon^{-(1-D)}$$

exists in $(0, \infty)$. 
For (2), a frequency of $\mathcal{L}$ is
\[ f = \sqrt{\lambda}/\pi = \frac{k}{l_j}. \]

The spectral zeta function of $\mathcal{L}$ is
\[ \zeta_{\nu}(s) = \sum_{j,k=1}^{\infty} (k \cdot l_j^{-1})^{-s} = \sum f w f^{-s} \]

Then
\[ \zeta_{\nu}(s) = \zeta_{\mathcal{L}}(s) \zeta (s) \]
relates spectral and geometric information.

For (3), we have the explicit (distributional) formula
\[
V(\varepsilon) = \sum_{\omega \in \mathcal{D}_\mathcal{L}} \text{res} \left( \frac{\zeta_{\mathcal{L}}(s)(2\varepsilon)^{1-s}}{s(1-s)}; \omega \right) + \mathcal{R}(\varepsilon) \\
= \sum_{\omega \in \mathcal{D}_\mathcal{L}} \left( \frac{\text{res} \left( \zeta_{\mathcal{L}}; \omega \right) 2^{1-\omega}}{\omega(1-\omega)} \right) \varepsilon^{1-\omega} + \mathcal{R}(\varepsilon).
\]

We want higher-dimensional analogues of these results.

Computing $V(\varepsilon)$ for the Koch curve provides
- a test of how well the theory holds up for $\Omega \subseteq \mathbb{R}^2$
- intuition about how to extend into $\mathbb{R}^2$
First, partition the $\varepsilon$-neighbourhood.

\begin{align*}
\varepsilon \in \left(3^{-(n+3/2)}, 3^{-(n+1/2)}\right) &= \left(3^{-(n+1)/\sqrt{3}}, 3^{-n/\sqrt{3}}\right).
\end{align*}

Figure 5. An approximation to the inner $\varepsilon$-neighbourhood of the Koch curve.

Figure 6. Another $\varepsilon$-neighbourhood of the Koch curve, for smaller $\varepsilon$. 
Count each type of piece:

<table>
<thead>
<tr>
<th>shape</th>
<th>number</th>
<th>volume (area)</th>
</tr>
</thead>
<tbody>
<tr>
<td>rectangles</td>
<td>$r_n = 4^n$</td>
<td>$\varepsilon 3^{-n}$</td>
</tr>
<tr>
<td>wedges</td>
<td>$w_n = \frac{2}{3}(4^n - 1)$</td>
<td>$\pi \varepsilon^2/6$</td>
</tr>
<tr>
<td>triangles</td>
<td>$u_n = \frac{2}{3}(4^n - 1) + 2$</td>
<td>$\varepsilon^2 \sqrt{3}/2$</td>
</tr>
<tr>
<td>fringe</td>
<td>$4^n$</td>
<td>$9^{1-n} \sqrt{3}/160$</td>
</tr>
</tbody>
</table>

A preliminary formula is

$$\hat{V}(\varepsilon) = \hat{V}_1(\varepsilon) + \hat{V}_2(\varepsilon) - \hat{V}_3(\varepsilon) + \hat{V}_4(\varepsilon),$$

where

$$\hat{V}_1(\varepsilon) := 4^n \cdot \varepsilon 3^{-n}$$

$$\hat{V}_2(\varepsilon) := \frac{2}{3}(4^n - 1) \cdot \frac{\pi \varepsilon^2}{6}$$

$$\hat{V}_3(\varepsilon) := \left(\frac{2}{3}(4^n - 1) + 2\right) \cdot \frac{\varepsilon^2 \sqrt{3}}{2}$$

$$\hat{V}_4(\varepsilon) := \left(\frac{4}{9}\right)^n \left(\frac{3^2 \sqrt{3}}{5 \cdot 2^5}\right)$$
Problem: formula contains a discrete variable. Need to convert to continuous:

\[ n = n(\varepsilon) = [x] = x - \{x\} \quad \text{for} \quad x = -\log_3(\varepsilon \sqrt{3}). \]

Figure 7. The exponent \( n = n(\varepsilon) \), as a function for \( \varepsilon \to 0 \).

Now as fn of \( \varepsilon, x = -\log_3(\varepsilon \sqrt{3}) \):

\[
\hat{V}(\varepsilon) = \varepsilon^{2-D} 4^{-\{x\}} \left( \frac{27 \sqrt{3}}{640} \{x\} + \frac{\sqrt{3}}{2} \{x\} + \left( \frac{\pi}{18} - \frac{\sqrt{3}}{6} \right) \right) - \varepsilon^2 \left( \frac{\pi}{9} + \frac{2\sqrt{3}}{9} \right)
\]
Convert to Fourier series using

\[ a^{-\{u\}} = \frac{a - 1}{a} \sum_{\beta \in \mathbb{Z}} \frac{e^{2\pi i \beta u}}{\log a + 2\pi i \beta} \]

and get

\[ \hat{V}(\varepsilon) = \varepsilon^{2-D} \sum_{n \in \mathbb{Z}} \left( -\frac{27\sqrt{3}}{2^9} \frac{e^{2\pi inx}}{\log 4/9+2\pi in} + \frac{\sqrt{3}}{8} \frac{e^{2\pi inx}}{\log 4/3+2\pi in} \right) + \left( \frac{3\pi}{72} - \frac{\sqrt{3}}{8} \right) \frac{e^{2\pi inx}}{\log 4+2\pi in} \right) - \varepsilon^2 \left( \frac{\pi}{9} + \frac{2\sqrt{3}}{3} \right). \]

Now: collect the error.

\textbf{Figure 8.} Where the error lies - the bold region is not within \( \varepsilon \) of \( K \).
We decompose an error block:

![Diagram showing decomposition of an error block into a central triangle and two wedge segments.](image)

Figure 9. \( w(\varepsilon) \) gives the width of the block

Use elementary methods to find the area \( A_1(\varepsilon) \):

![Diagram showing the height of the central triangle.](image)

Figure 10. Finding the height of the central triangle

\[
A_1(\varepsilon) = \varepsilon \frac{w}{3} - \varepsilon^2 \sin^{-1} \left( \frac{w}{6\varepsilon} \right) - \varepsilon \frac{w}{6} \sqrt{1 - \left( \frac{w}{6\varepsilon} \right)^2}.
\]
The area of the error in this block is

\[
B(\varepsilon) := \sum_{k=1}^{\infty} 2^{k-1} \left( \frac{\varepsilon w(\varepsilon)}{3^k} - \varepsilon^2 \sin^{-1} \left( \frac{w(\varepsilon)}{2 \cdot 3^k \varepsilon} \right) \right) - \varepsilon \frac{w(\varepsilon)}{2 \cdot 3^k} \sqrt{1 - \left( \frac{w(\varepsilon)}{2 \cdot 3^k \varepsilon} \right)^2}. \]

Figure 11. Naming the trianglets

Apply series expansions for \(\sin^{-1} u\) and \(\sqrt{1 - u^2}\),

\[
w(\varepsilon) = 3^{-[x]} = 3^{\{x\}} \varepsilon \sqrt{3},
\]

and Fubini Thm, to get
\[ B(\varepsilon) = \frac{\varepsilon w(\varepsilon)}{2} \]

\[ + \sum_{m=0}^{\infty} \frac{(2m)!}{2^{4m+1} (m!)^2} \left( \frac{w(\varepsilon)^2}{2^3 \varepsilon (m+1)} \cdot \frac{1}{3^{2m+3} - 2} \right) \left( \frac{w(\varepsilon)}{\varepsilon} \right)^{2m+1} \]

How many such error blocks are there?

Some blocks are whole; others form as \( \varepsilon \to 0 \).

\textbf{Figure 12.} Error block formation
We count the error blocks

\[ \delta(\varepsilon) = c_n + p_n h \]

\[ = c(\varepsilon) + p(\varepsilon) h(\varepsilon) \]

\[ = \frac{\varepsilon^{-D}}{6} 4^{-\{x\}} + \frac{\varepsilon^{-D}}{6} 4^{-\{x\}} h(\varepsilon) + \frac{2}{3} h(\varepsilon) - \frac{4}{3} \]

where \( h(\varepsilon) \) is some function indicating what portion of the partial block has formed.

\[ 0 \leq h(\varepsilon) = h \left( \frac{\varepsilon}{3} \right) \leq \mu < 1 \]

We don’t know \( h(\varepsilon) \) explicitly, but we do know

\[ h(\varepsilon) = \sum_{\alpha \in \mathbb{Z}} g_\alpha e^{2\pi i \alpha x} \]

Total error is

\[ E(\varepsilon) = \delta(\varepsilon) B(\varepsilon) \]

Compute the desired volume formula as

\[ V(\varepsilon) = \hat{V}(\varepsilon) - E(\varepsilon) \]

by converting everything into series expansion.
After 11 pages of calculations . . .

\[ V(\varepsilon) = G_1(\varepsilon)\varepsilon^{2-D} + G_2(\varepsilon)\varepsilon^2, \]

where

\[
G_1(\varepsilon) := \frac{1}{\log 3} \sum_{n \in \mathbb{Z}} \left( a_n + \sum_{\nu \in \mathbb{Z}} b_{\nu} g_{n-\nu} \right) (-1)^n \varepsilon^{-in\mathbf{p}}
\]

\[
G_2(\varepsilon) := \frac{1}{\log 3} \sum_{n \in \mathbb{Z}} \left( \sigma_n + \sum_{\nu \in \mathbb{Z}} \tau_{\nu} g_{n-\nu} \right) (-1)^n \varepsilon^{-in\mathbf{p}},
\]

are periodic functions of multiplicative period 3, and

\[
a_n = -\frac{3^{9/2}}{2^9(D-2+in\mathbf{p})} + \frac{3^{3/2}}{2^3(D-1+in\mathbf{p})} + \frac{\pi-3^{3/2}}{2^3(D+in\mathbf{p})} - \frac{1}{2} b_n,
\]

\[
b_n = \sum_{m=1}^{\infty} \frac{3^{m-1/2}(2m-2)!}{2^{4m+1}(2m+1)m!(m-1)!} \cdot \frac{(4-3^{2m+1})}{(3^{2m+1}-2)(D-2m-1+iv\mathbf{p})},
\]

\[
\sigma_n = \left( \frac{\pi}{9} + \frac{2\sqrt{3}}{3} \right) \delta_0^n - \tau_n,
\]

\[
\tau_n = \sum_{m=1}^{\infty} \frac{(2m)!3^{m+1/2}}{2^{4m-2}(2m+1)m!(m-1)!} \cdot \frac{1-3^{2m+1}}{(3^{2m+1}-2)(-2m-1+iv\mathbf{p})}.
\]

The \( g_\alpha \) are Fourier coefficients of function which counts the error blocks (actually, it describes how much has formed).
However, the coefficients are not the interesting part. The formula for $V(\varepsilon)$ contains all the complex dimensions! We rewrite as

$$V(\varepsilon) = \sum_{n \in \mathbb{Z}} G_3(\varepsilon)\varepsilon^{2-D-inp} + \sum_{n \in \mathbb{Z}} G_4(\varepsilon)\varepsilon^{2-inp}.$$ 

This gives the possible dimensions

$$\mathcal{D}_{\partial\Omega} = \{D + inp : n \in \mathbb{Z}\} \cup \{inp : n \in \mathbb{Z}\}.$$
Notes on $h(\varepsilon)$:

The least upper bound of $h(\varepsilon)$ is $\mu = C(\varepsilon)/B(\varepsilon)$:

![Graph of C(\varepsilon) and B(\varepsilon)](image)

**Figure 13.** $\mu$ is the ratio $C(\varepsilon)/B(\varepsilon)$.

Three essential properties:

(i) $h(\varepsilon)$ oscillates multiplicatively,

(ii) $h(\varepsilon_k) = \lim_{\vartheta \to 0^-} h(\varepsilon_k + \vartheta) = 0$,

(iii) $\lim_{\vartheta \to 0^+} h(\varepsilon_k + \vartheta) = \mu$,

where $\varepsilon_k = \frac{3^{-k}}{\sqrt{3}}$. Compare to

$$\tilde{h}(\varepsilon) = \mu \cdot \{-[x] - x\}$$

![Graph of h(\varepsilon) and \tilde{h}(\varepsilon)](image)

**Figure 14.** The Cantor-like function $h$ and the approximation $\tilde{h}$. 

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