A TUBE FORMULA FOR THE KOCH SNOWFLAKE CURVE, WITH APPLICATIONS TO COMPLEX DIMENSIONS.

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Current (i.e., unfinished) draught of the full version is available at
`http://math.ucr.edu/~epearse/koch.pdf`.

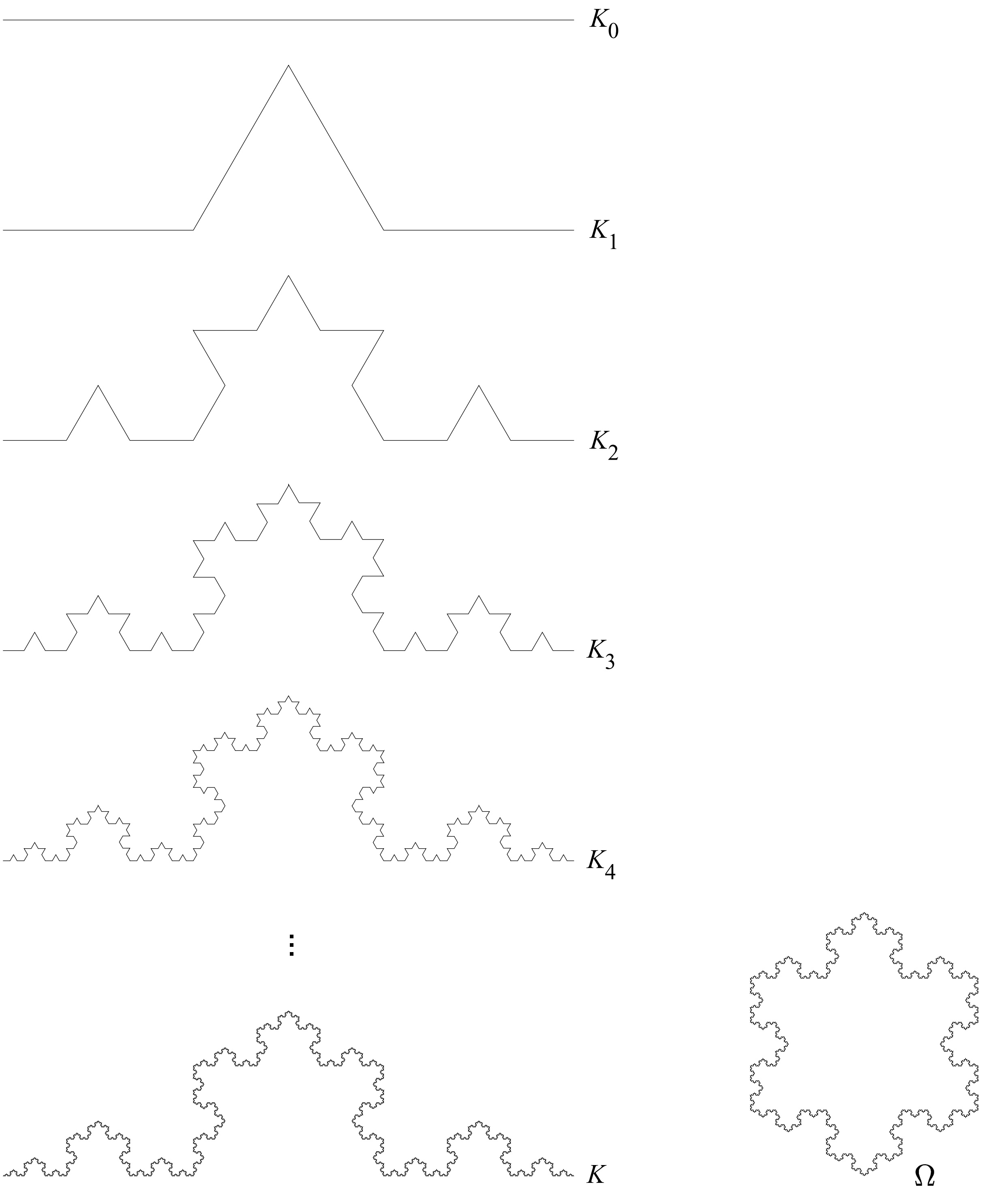


FIGURE 1. The Koch curve K (left) and the Koch snowflake Ω (right).

Goal: derive a formula for the ε-neighbourhood of the Koch curve (and snowflake).

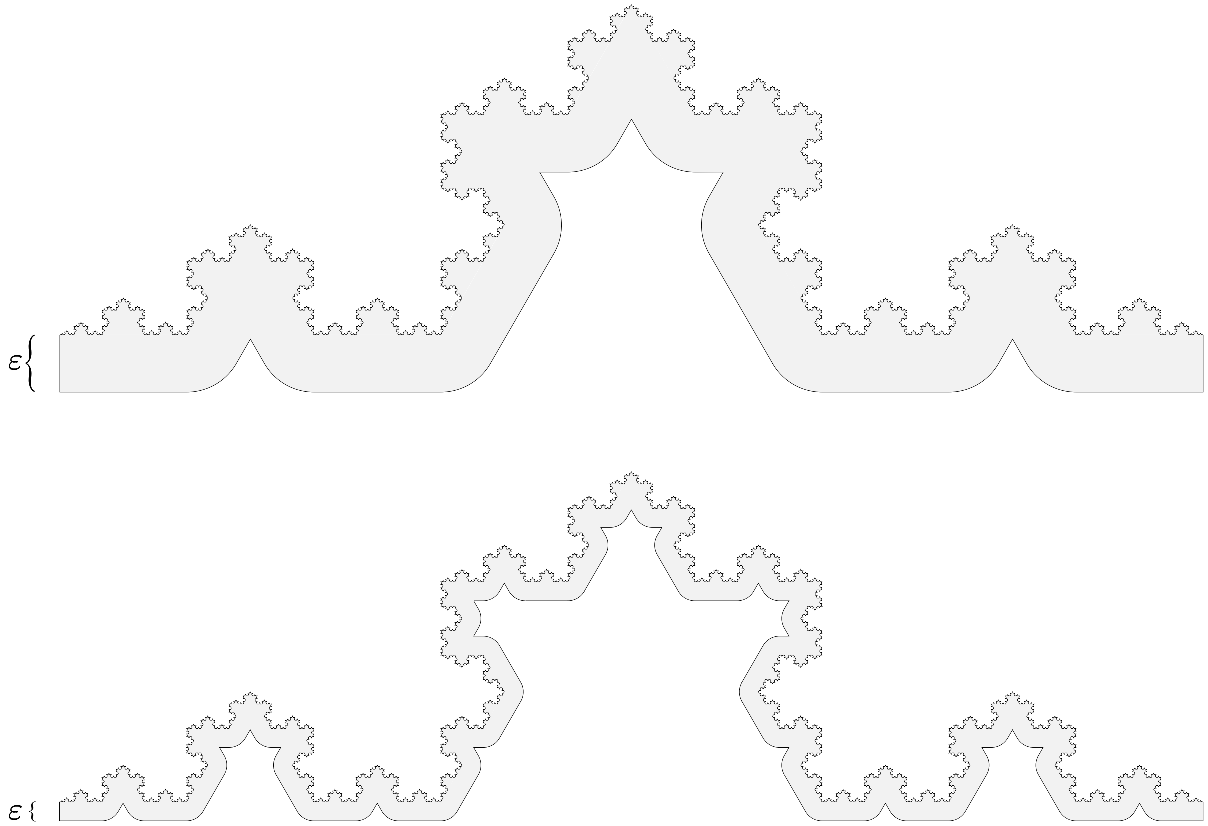


FIGURE 2. The ε-neighbourhood of the Koch curve, for two different values of ε.

We want to find a formula for

$$\begin{aligned} V(\varepsilon) &= \text{area of shaded region} \\ &= \mathrm{vol}_2\{x \in \Omega : d(x, \partial\Omega) < \varepsilon\} \end{aligned}$$

Q: What use is $V(\varepsilon)$?

A: A precise formula for $V(\varepsilon)$ will help towards extending the theory of fractal strings into higher dimensions.

A *fractal string* is any bounded open subset of $\mathbb{R}$

$$\mathcal{L} := \{l_j\}_{j=1}^{\infty}, \quad \text{with } \sum_{j=1}^{\infty} l_j < \infty.$$

$$l_1 \geq l_2 \geq l_3 \geq \dots,$$

or distinctly (with multiplicity):

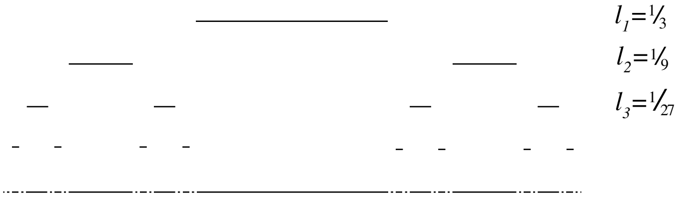
$$l_1 > l_2 > l_3 > \dots.$$

Idea/origin: comes from studying fractal subsets

$$\partial\mathcal{L} \subseteq \mathbb{R}.$$

Figure 3. The Cantor Set

Figure 4. The Cantor String

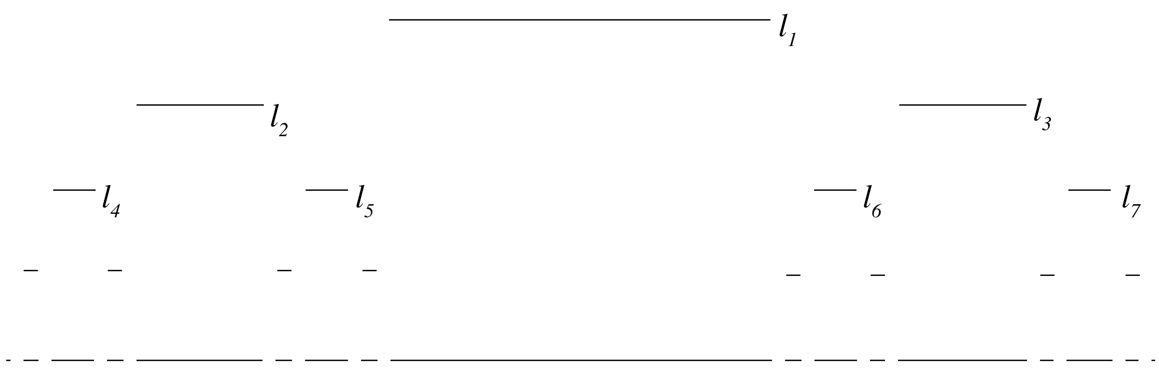


The Cantor String example has lengths

$$\left\{3^{-(n+1)}\right\}$$

with multiplicities

$$w_{3^{-(n+1)}} = 2^n.$$



$$\mathcal{CS} = \left\{\tfrac{1}{3}, \tfrac{1}{9}, \tfrac{1}{9}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \tfrac{1}{27}, \dots\right\}$$

The *geometric zeta function* of a string

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s = \sum_l w_l l^s$$

encodes all this information.

Example:

$$\zeta_{\mathcal{CS}}(s) = \sum_{n=0}^{\infty} 2^n 3^{-(n+1)s} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}.$$

Three key things about $\zeta_{\mathcal{L}}$:

(1) Relates to the dimension of $\partial\mathcal{L}$.
(2) Connects spectral and geometric properties.
(3) Gives an explicit formula for $V(\varepsilon)$.

For (1), recover the Minkowski dimension

$$\begin{aligned} D_{\partial\mathcal{L}} &:= \inf\{t \geq 0 : V(\varepsilon) = O\left(\varepsilon^{1-t}\right) \text{ as } \varepsilon \to 0^+\} \\ &= \inf\{\sigma \geq 0 : \zeta_{\mathcal{L}}(s) < \infty\} \end{aligned}$$

Generalize and define the *complex dimensions*:

$$\mathcal{D} = \{\omega \in \mathbb{C} : \zeta_{\mathcal{L}} \text{ has a pole at } \omega\}$$

Theorem 1 (Structure of Complex Dimensions)**.** *If $\partial\mathcal{L}$ is self-similar, then either*

(1) $\mathcal{D}$ is "periodic" and $\partial\mathcal{L}$ is not measurable, or
(2) $\mathcal{D}$ is "quasiperiodic" and $\partial\mathcal{L}$ is measurable.

Here, $\partial\mathcal{L}$ is *(Minkowski) measurable* iff

$$\mathcal{M} = \mathcal{M}(D; \partial\mathcal{L}) = \lim_{\varepsilon \to 0+} V(\varepsilon)\varepsilon^{-(1-D)}$$

exists in $(0, \infty)$.

For (2), a *frequency* of $\mathcal{L}$ is

$$f = \sqrt{\lambda}/\pi = \frac{k}{l_j}.$$

The *spectral zeta function* of $\mathcal{L}$ is

$$\zeta_\nu(s) = \sum_{j,k=1}^{\infty} (k \cdot l_j^{-1})^{-s} = \sum_f w_f f^{-s}$$

Then

$$\zeta_\nu(s) = \zeta_{\mathcal{L}}(s)\zeta(s)$$

relates spectral and geometric information.

For (3), we have the explicit (distributional) formula

$$\begin{aligned} V(\varepsilon) &= \sum_{\omega \in \mathcal{D}_{\mathcal{L}}} \operatorname{res}\left(\frac{\zeta_{\mathcal{L}}(s)(2\varepsilon)^{1-s}}{s(1-s)}; \omega \right) + \mathcal{R}(\varepsilon) \\ &= \sum_{\omega \in \mathcal{D}_{\mathcal{L}}} \left(\frac{\operatorname{res}\left(\zeta_{\mathcal{L}}; \omega\right) 2^{1-\omega}}{\omega(1-\omega)} \right) \varepsilon^{1-\omega} + \mathcal{R}(\varepsilon). \end{aligned}$$

We want higher-dimensional analogues of these results.

Computing $V(\varepsilon)$ for the Koch curve provides

- a test of how well the theory holds up for $\Omega \subseteq \mathbb{R}^2$
- intuition about how to extend into $\mathbb{R}^2$

First, partition the ε-neighbourhood.

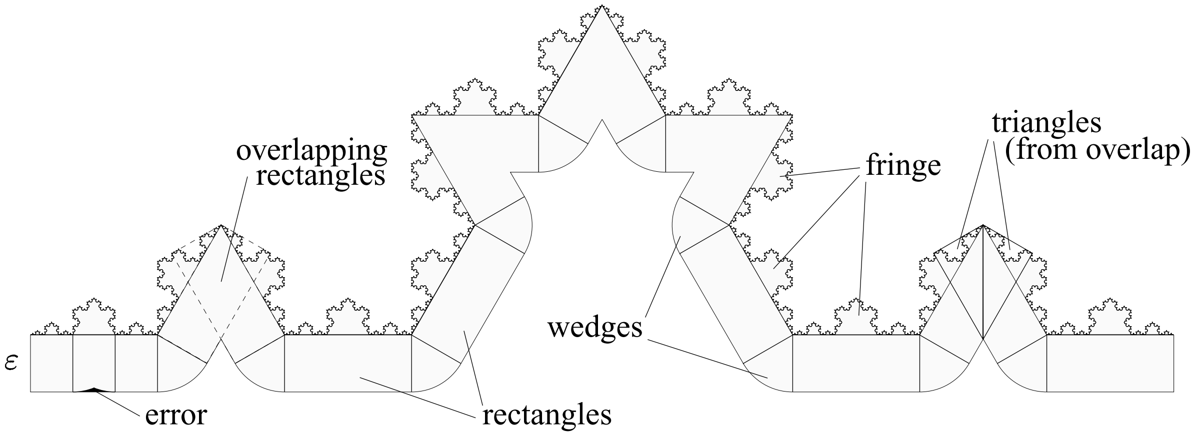


FIGURE 5. An approximation to the inner ε-neighbourhood of the Koch curve.

The level of refinement is based on

$$\varepsilon \in \left(3^{-(n+3/2)}, 3^{-(n+1/2)}\right] = \left(3^{-(n+1)}/\sqrt{3}, 3^{-n}/\sqrt{3}\right].$$

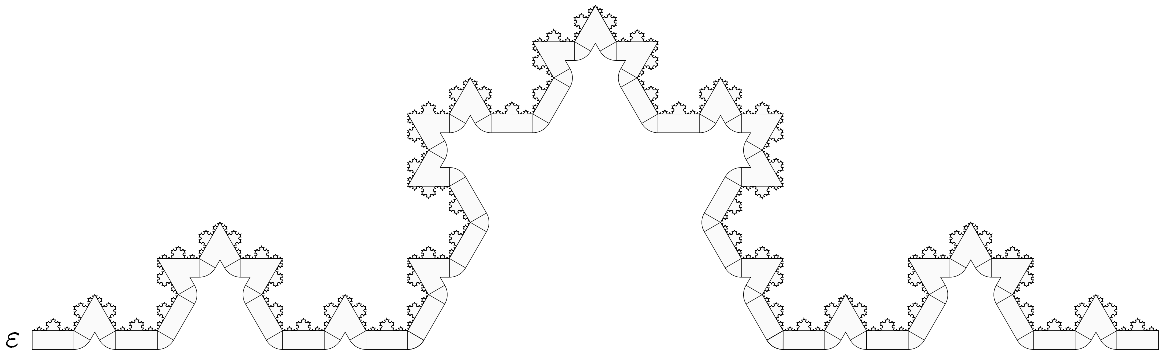
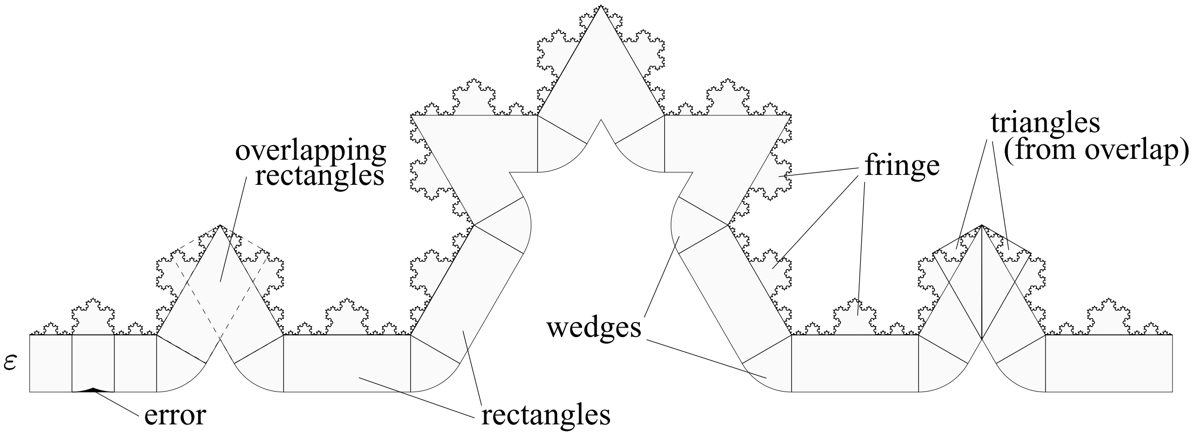


FIGURE 6. Another ε-neighbourhood of the Koch curve, for smaller ε.



Count each type of piece:

shape	number	volume(area)
rectangles	$r_n = 4^n$	$\varepsilon 3^{-n}$
wedges	$w_n = \frac{2}{3}(4^n - 1)$	$\pi\varepsilon^2/6$
triangles	$u_n = \frac{2}{3}(4^n - 1) + 2$	$\varepsilon^2\sqrt{3}/2$
fringe	4^n	$9^{1-n}\sqrt{3}/160$

A preliminary formula is

$$\hat{V}(\varepsilon) = \hat{V}_1(\varepsilon) + \hat{V}_2(\varepsilon) - \hat{V}_3(\varepsilon) + \hat{V}_4(\varepsilon),$$

where

$$\hat{V}_1(\varepsilon) := 4^n \cdot \varepsilon 3^{-n}$$

$$\hat{V}_2(\varepsilon) := \frac{2}{3}(4^n - 1) \cdot \frac{\pi\varepsilon^2}{6}$$

$$\hat{V}_3(\varepsilon) := \left(\frac{2}{3}(4^n - 1) + 2\right) \cdot \frac{\varepsilon^2\sqrt{3}}{2}$$

$$\hat{V}_4(\varepsilon) := \left(\frac{4}{9}\right)^n \left(\frac{3^2\sqrt{3}}{5 \cdot 2^5}\right)$$

Problem: formula contains a discrete variable.
Need to convert to continuous:

$$n = n(\varepsilon) = [x] = x - \{x\} \quad \text{for} \quad x = -\log_3(\varepsilon\sqrt{3}).$$

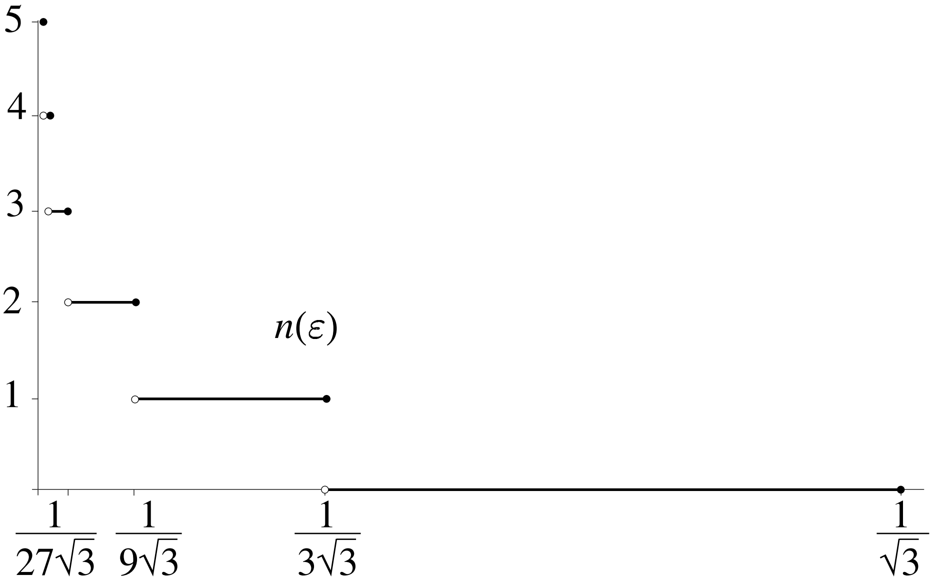


Figure 7. The exponent $n = n(\varepsilon)$, as a function for $\varepsilon \to 0$.

Now as fn of $\varepsilon, x = -\log_3(\varepsilon\sqrt{3})$:

$$\hat{V}(\varepsilon) = \varepsilon^{2-D} 4^{-\{x\}} \left(\frac{27\sqrt{3}}{640} 9^{\{x\}} + \frac{\sqrt{3}}{2} 3^{\{x\}} + \left(\frac{\pi}{18} - \frac{\sqrt{3}}{6} \right) \right) - \varepsilon^2 \left(\frac{\pi}{9} + \frac{2\sqrt{3}}{9} \right)$$

Convert to Fourier series using

$$a^{-\{u\}} = \frac{a-1}{a} \sum_{\beta \in \mathbb{Z}} \frac{e^{2\pi i \beta u}}{\log a + 2\pi i \beta}$$

and get

$$\hat{V}(\varepsilon) = \varepsilon^{2-D} \sum_{n \in \mathbb{Z}} \left(-\frac{27\sqrt{3}}{2^9} \frac{e^{2\pi i n x}}{\log 4/9 + 2\pi i n} + \frac{\sqrt{3}}{8} \frac{e^{2\pi i n x}}{\log 4/3 + 2\pi i n} \right.$$
$$\left. + \left(\frac{3\pi}{72} - \frac{\sqrt{3}}{8} \right) \frac{e^{2\pi i n x}}{\log 4 + 2\pi i n} \right) - \varepsilon^2 \left(\frac{\pi}{9} + \frac{2\sqrt{3}}{3} \right).$$

Now: collect the error.

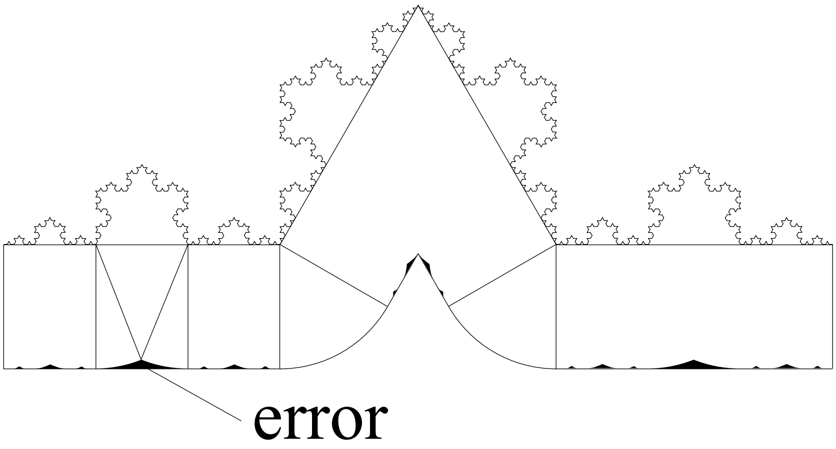


Figure 8. Where the error lies - the bold region is not within ε of K.

We decompose an error block:

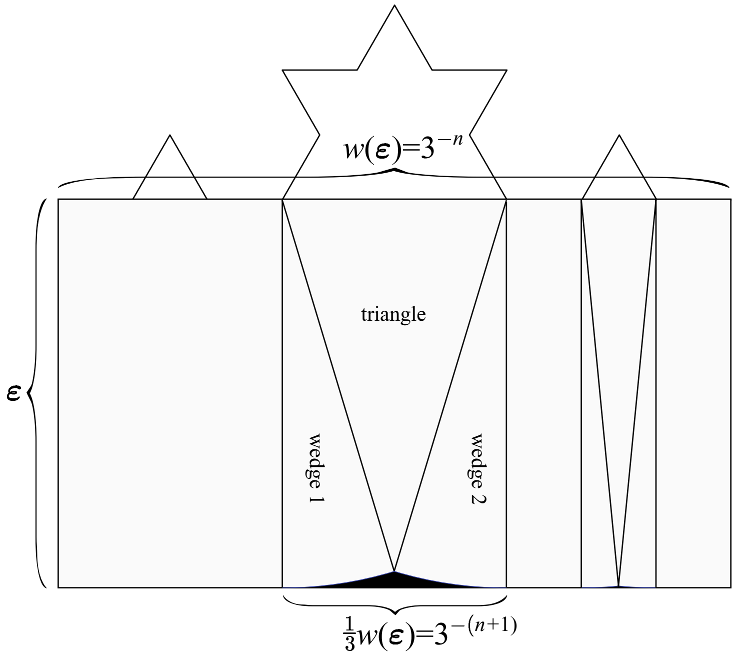


FIGURE 9. $w(\varepsilon)$ gives the width of the block

Use elementary methods to find the area $A_1(\varepsilon)$:

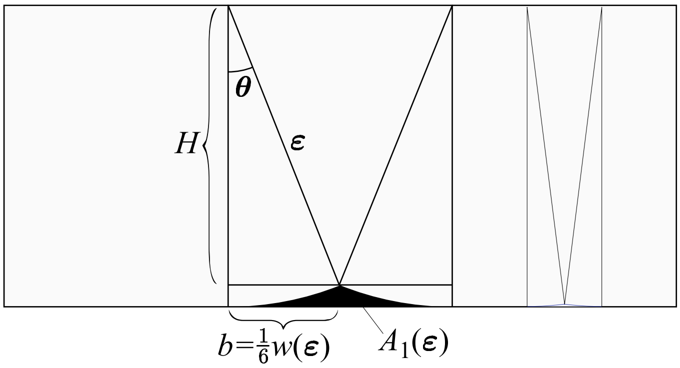


FIGURE 10. Finding the height of the central triangle

$$A_1(\varepsilon) = \varepsilon\frac{w}{3} - \varepsilon^2 \sin^{-1}\left(\frac{w}{6\varepsilon}\right) - \varepsilon\frac{w}{6}\sqrt{1-\left(\frac{w}{6\varepsilon}\right)^2}.$$

The area of the error in this block is

$$B(\varepsilon) := \sum_{k=1}^{\infty} 2^{k-1} \left(\varepsilon \frac{w(\varepsilon)}{3^k} - \varepsilon^2 \sin^{-1}\left(\frac{w(\varepsilon)}{2\cdot 3^k \varepsilon}\right) - \varepsilon \frac{w(\varepsilon)}{2\cdot 3^k} \sqrt{1 - \left(\frac{w(\varepsilon)}{2\cdot 3^k \varepsilon}\right)^2} \right).$$

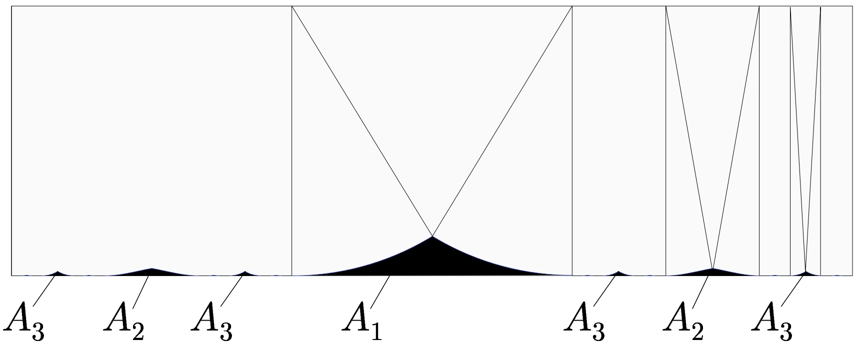


Figure 11. Naming the trianglets

Apply series expansions for $\sin^{-1} u$ and $\sqrt{1-u^2}$,

$$w(\varepsilon) = 3^{-[x]} = 3^{\{x\}}\varepsilon\sqrt{3},$$

and Fubini Thm, to get

$$
\begin{aligned}
B(\varepsilon) = {} & \frac{\varepsilon w(\varepsilon)}{2} \\
& + \sum_{m=0}^{\infty} \frac{(2m)!}{2^{4m+1}(m!)^2} \left(\frac{w(\varepsilon)^2}{2^3\varepsilon(m+1)} \cdot \frac{1}{3^{2m+3}-2} \right. \\
& \left. \qquad - \frac{\varepsilon^2}{2m+1} \cdot \frac{1}{3^{2m+1}-2} \right) \left(\frac{w(\varepsilon)}{\varepsilon} \right)^{2m+1}.
\end{aligned}
$$

How many such error blocks are there?

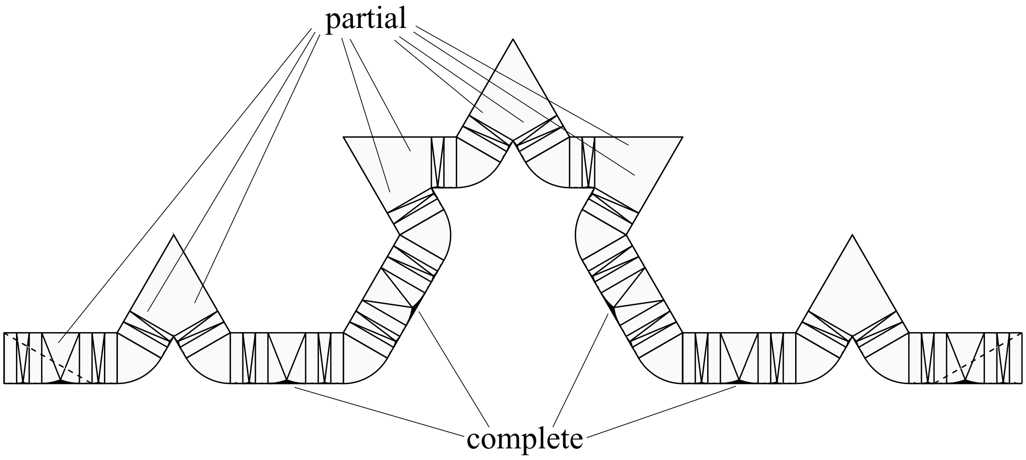


FIGURE 12. Error block formation

Some blocks are whole; others form as $\varepsilon \to 0$.

We count the error blocks

$$\begin{aligned}\delta(\varepsilon) &= c_n + p_n h \\ &= c(\varepsilon) + p(\varepsilon) h(\varepsilon) \\ &= \tfrac{\varepsilon^{-D}}{6} 4^{-\{x\}} + \tfrac{\varepsilon^{-D}}{6} 4^{-\{x\}} h(\varepsilon) + \tfrac{2}{3} h(\varepsilon) - \tfrac{4}{3}\end{aligned}$$

where $h(\varepsilon)$ is some function indicating what portion of the partial block has formed.

$$0 \leq h(\varepsilon) = h\left(\frac{\varepsilon}{3}\right) \leq \mu < 1$$

We don't know $h(\varepsilon)$ explicitly, but we do know

$$h(\varepsilon) = \sum_{\alpha \in \mathbb{Z}} g_\alpha e^{2\pi i \alpha x}$$

Total error is

$$E(\varepsilon) = \delta(\varepsilon) B(\varepsilon)$$

Compute the desired volume formula as

$$V(\varepsilon) = \hat{V}(\varepsilon) - E(\varepsilon)$$

by converting everything into series expansion.

After 11 pages of calculations …

$$V(\varepsilon) = G_1(\varepsilon)\varepsilon^{2-D} + G_2(\varepsilon)\varepsilon^2,$$

where

$$G_1(\varepsilon) := \frac{1}{\log 3} \sum_{n\in\mathbb{Z}} \left(a_n + \sum_{\nu\in\mathbb{Z}} b_\nu g_{n-\nu} \right) (-1)^n \varepsilon^{-in\mathbf{p}}$$

$$G_2(\varepsilon) := \frac{1}{\log 3} \sum_{n\in\mathbb{Z}} \left(\sigma_n + \sum_{\nu\in\mathbb{Z}} \tau_\nu g_{n-\nu} \right) (-1)^n \varepsilon^{-in\mathbf{p}},$$

are periodic functions of multiplicative period 3, and

$$a_n = -\tfrac{3^{9/2}}{2^9(D-2+in\mathbf{p})} + \tfrac{3^{3/2}}{2^3(D-1+in\mathbf{p})} + \tfrac{\pi-3^{3/2}}{2^3(D+in\mathbf{p})} - \tfrac{1}{2}b_n,$$

$$b_n = \sum_{m=1}^{\infty} \tfrac{3^{m-1/2}(2m-2)!}{2^{4m+1}(2m+1)m!(m-1)!} \cdot \tfrac{(4-3^{2m+1})}{(3^{2m+1}-2)(D-2m-1+i\nu\mathbf{p})},$$

$$\sigma_n = \left(\tfrac{\pi}{9} + \tfrac{2\sqrt{3}}{3} \right) \delta_0^n - \tau_n,$$

$$\tau_n = \sum_{m=1}^{\infty} \tfrac{(2m)!3^{m+1/2}}{2^{4m-2}(2m+1)m!(m-1)!} \cdot \tfrac{1-3^{2m+1}}{(3^{2m+1}-2)(-2m-1+i\nu\mathbf{p})}.$$

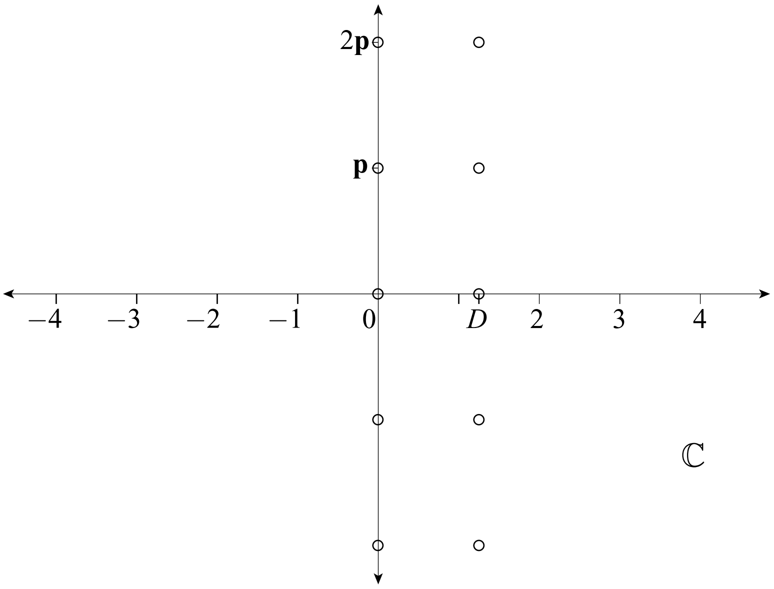
The g_α are Fourier coefficients of function which counts the error blocks (actually, it describes how much has formed).

However, the coefficients are not the interesting part. The formula for $V(\varepsilon)$ contains all the complex dimensions! We rewrite as

$$V(\varepsilon) = \sum_{n\in\mathbb{Z}} G_3(\varepsilon)\varepsilon^{2-D-in\mathbf{p}} + \sum_{n\in\mathbb{Z}} G_4(\varepsilon)\varepsilon^{2-in\mathbf{p}}.$$

This gives the possible dimensions

$$\mathcal{D}_{\partial\Omega} = \{D + in\mathbf{p} : n \in \mathbb{Z}\} \cup \{in\mathbf{p} : n \in \mathbb{Z}\}.$$



Notes on $h(\varepsilon)$:

The least upper bound of $h(\varepsilon)$ is $\mu = C(\varepsilon)/B(\varepsilon)$:

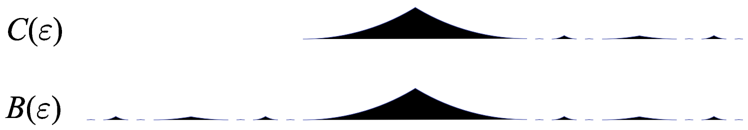


FIGURE 13. μ is the ratio $C(\varepsilon)/B(\varepsilon)$.

Three essential properties:

(i) $h(\varepsilon)$ oscillates multiplicatively,

(ii) $h(\varepsilon_k) = \lim_{\vartheta \to 0^-} h(\varepsilon_k + \vartheta) = 0$,

(iii) $\lim_{\vartheta \to 0^+} h(\varepsilon_k + \vartheta) = \mu$,

where $\varepsilon_k = \frac{3^{-k}}{\sqrt{3}}$. Compare to

$$\tilde{h}(\varepsilon) = \mu \cdot \{-[x] - x\}$$

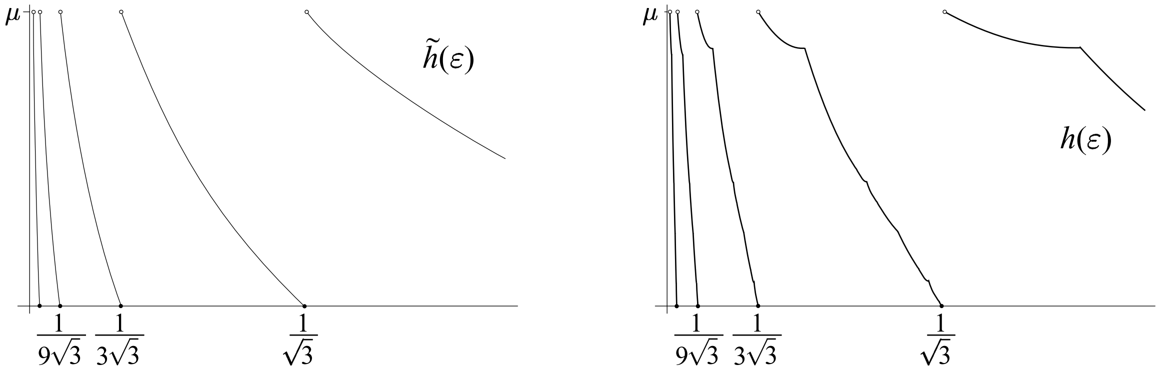


FIGURE 14. The Cantor-like function h and the approximation $\tilde{h}$.