From *Théorie des Distributions* by Laurent Schwartz.

II. Differentiation

II.2. **Examples of differentiation. The case of one variable ($n = 1$).**

II.2.3. *Pseudofunctions. Hadamard finite part.*
We calculate the derivative of a function $f(x)$ which is equal to 0 for $x < 0$ and $\frac{1}{\sqrt{x}}$ for $x > 0$ (not defined for $x = 0$).

This derivative is surely 0 on the interval $(-\infty, 0)$ and equal to the function $-\frac{1}{2}x^{3/2}$ in the interval $(0, \infty)$ because in each of these two open intervals, $f(x)$ is a continuous function with continuous derivatives (in the usual sense).

$$\begin{cases} f'(\varphi) & = -f(\varphi') = -\int_0^\infty \varphi'(x)x^{-1/2}dx \\ & = -\lim_{\varepsilon\to 0}\int_0^\infty \varphi'(x)x^{-1/2}dx \end{cases} \tag{II.2.9}$$

Integrating by parts:

$$f'(\varphi) = \lim_{\varepsilon\to 0}\left[\frac{\varphi(\varepsilon)}{\sqrt{\varepsilon}} + \int_\varepsilon^\infty \varphi(x)\left(\frac{1}{2}x^{-3/2}\right)dx\right]. \tag{II.2.10}$$

As $\varphi(\varepsilon) = \varphi(0) + O(\varepsilon)$ for $\varepsilon \to 0$, one has finally

$$f'(\varphi) = \lim_{\varepsilon\to 0}\left[\int_\varepsilon^\infty \varphi(x)\left(\frac{1}{2}x^{-3/2}\right)dx + \varphi(0)\varepsilon^{-1/2}\right]. \tag{II.2.11}$$

We find here an idea introduced in [**?**, pages 184-215] in aid of the theory of PDE: the finite part of a divergent integral.

Let g be a summable function on every interval $(a+\varepsilon, b)$, $\varepsilon > 0$, but not summable on (a, b). It could happen that $g(x)$ could be the sum of a polynomial in $\frac{1}{x-a}$ and a function $h(x)$ which is summable in (a, b):

$$g(x) = P\left(\frac{1}{x-a}\right) + h(x) = \sum \frac{A_\nu}{(x-a)^{\lambda_\nu}} + h(x). \tag{II.2.12}$$

We mean the word polynomial in the following sense: a sum of monomials with any complex exponents λ_ν that have $\mathrm{Re}(\lambda_\nu) \geq 1$. *We must also assume initially that these exponents are not integers.* So we see that one can write

$$\int_{a+\varepsilon}^b g(x)\,dx = I(\varepsilon) + F(\varepsilon). \tag{II.2.13}$$

$I(\varepsilon)$, the infinite part of the integral, is a polynomial in $\frac{1}{\varepsilon}$; a sum of monomials with nonintegral complex exponents

$$I(\varepsilon) = \sum \frac{A_\nu}{\lambda_\nu - 1}\left(\frac{1}{\varepsilon}\right)^{\lambda_\nu - 1}, \tag{II.2.14}$$

and $F(\varepsilon)$ has a finite limit F as $\varepsilon \to 0$. It's this quantity F that M. Hadamard calls the finite part of the integral $\int_a^b g(x)\,dx$ and we note

$$F = \mathrm{Fp.}\int_a^b g(x)\,dx = -\sum \frac{A_\nu}{\lambda_\nu - 1}\left(\frac{1}{b-a}\right)^{\lambda_\nu - 1} + \int_a^b h(x)\,dx. \tag{II.2.15}$$

The principal properties of such an integral generalize to the following:

1. The definition is invariant under a change of variables. If $x = x(t), t = t(x)$, is a C^∞ homeomorphism, one has

$$\mathrm{Fp.}\int_a^b g(x)\,dx = \mathrm{Fp.}\int_{t(a)}^{t(b)} g[x(t)]x'(t)\,dt. \tag{II.2.16}$$

2. We compute the integral $\int_a^b g(x)(x-a)^\lambda dx$. Unless λ is a complex number with positive real part sufficiently large, this is just the ordinary integral of a summable function.

$$\begin{aligned} F(\lambda) &= \int_a^b g(x)(x-a)^\lambda\,dx \\ &= -\sum \frac{A_\nu}{\lambda_\nu - \lambda - 1}\left(\frac{1}{b-a}\right)^{\lambda_\nu - \lambda - 1} + \int_a^b h(x)(x-a)^\lambda\,dx. \end{aligned} \tag{II.2.17}$$

The first term may be extended via analytic continuation; it's a meromorphic function of λ on the entire complex plane with a finite number of poles $\lambda = \lambda_\nu - 1$. The second term is holomorphic for $\mathrm{Re}(\lambda) > 0$ and continuous as $\lambda \to 0$.

Thus $F(\lambda)$ is meromorphic for $\mathrm{Re}(\lambda) > 0$; like the noninteger λ_ν, it is continuous for $\lambda \to 0$ and has limit

$$F(0) = -\sum \frac{A_\nu}{\lambda_\nu - 1}\left(\frac{1}{b-a}\right)^{\lambda_\nu - 1} + \int_a^b h(x)\,dx = \mathrm{Fp.}\int_a^b g(x)\,dx. \tag{II.2.18}$$

The finite part of an integral thus appears to be the analytic continuation of an ordinary integral.

3. If $\varphi(x)$ is infinitely differential, the function $g(x)\varphi(x)$ has analogous properties to g on (a,b) and one can define the finite part $\int_a^b g(x)\varphi(x)dx$. One can easily see that this is a *continuous linear functional* on $\varphi \in \mathcal{D}$. Thus $g(x)$, although not summable on (a,b), defines a distribution which we call a *pseudofunction* and denote by $\mathrm{Pf.}g$.

$$\mathrm{Pf.}g(\varphi) = \mathrm{Fp.}\int_a^b g(x)\varphi(x)\,dx. \tag{II.2.19}$$

All that we have come to say about a function g which vanishes outside a finite interval (a,b) and is singular toward a extends to functions g which are defined on all of $\mathbb{R}$. If g is a function which is summable over every compact set, save but in the neighborhood of certain points a_l, of which there are only finitely many in any bounded interval; and if in each such neighborhood (to the left and to the right of a_l) g is the sum of a polynomial in $\frac{1}{|x-a_l|}$ with complex noninteger exponents and a summable function (this function need not be the same to the right and to the left of a_l), then one can define unambiguously the integral

$$\text{Pf.}g(\varphi) = \text{Fp.}\int_{-\infty}^{\infty} g(x)\varphi(x)\,dx, \quad \varphi \in \mathcal{D} \tag{II.2.20}$$

(this integral is computed as a sum of integrals extended to finite intervals where g is not singular except at the endpoints).

We will see that $g(x)$ is the function $[f']$, the derivative (in the usual sense) of the function f defined on page 38,

$$[f'] = \begin{cases} 0, & x < 0, \\ -\frac{1}{2}x^{-3/2}, & x > 0, \end{cases}$$

the distributional derivative f' defined by the formula (II.2.11) is none other than the pseudofunction Pf.$[f']$.

$$f'(\varphi) = \text{Fp.}\int_{-\infty}^{\infty} \varphi(x)\left(-\tfrac{1}{2}x^{-3/2}\right)\,dx = \text{Pf.}[f'](\varphi). \tag{II.2.21}$$

Remark. The function $[f']$ is nonpositive, in the usual sense of the word. But the distribution pseudofunction $f' = \text{Pf.}[f']$ is not at all a nonpositive distribution. One doesn't necessarily have $f'(\varphi) \geq 0$ for $\varphi \geq 0$. Besides, f' is not a nonpositive measure since $[f']$ is not summable in a neighborhood of the origin. In contrast, f' is a well-defined nonpositive distribution (nonpositive measure) in the open set $\Omega = \mathbb{R}\backslash\{0\}$. Thus f', defined in $\mathbb{R}$, is nonpositive in Ω; it is equal in Ω to a nonpositive measure defined in Ω which is not extendable to a measure on $\mathbb{R}$.

We will now see now what happens with certain integer exponents λ_ν.

1. We always put

$$g(x) = \sum_\nu \frac{A_\nu}{(x-a)^{\lambda_\nu}} + h(x) = \sum_{\nu\neq 1} \frac{A_\nu}{(x-a)^{\lambda_\nu}} + \frac{A_1}{x-a} + h(x). \tag{II.2.22}$$

So we must also take

$$I(\varepsilon) = \sum_{\nu\neq 1} \frac{A_\nu}{\lambda_\nu - 1}\left(\frac{1}{\varepsilon}\right)^{\lambda_\nu - 1} + A_1 \log\frac{1}{\varepsilon} \tag{II.2.23}$$

and

$$F = \text{Fp.}\int_a^b g(x)\,dx = -\sum_{\nu\neq 1} \frac{A_\nu}{\lambda_\nu - 1}\left(\frac{1}{b-a}\right)^{\lambda_\nu - 1} + A_1\log(b-a) + \int_a^b h(x)\,dx. \tag{II.2.24}$$

Thus $I(\varepsilon)$ is no longer a polynomial; it's the sum of a polynomial in $\frac{1}{\varepsilon}$ with complex exponents which may be integers (but not 0) and some logarithmic term.

2. The finite part is no longer invariant under change of variables. Thus

$$\text{Fp.} \int_0^1 \frac{dx}{x} = 0; \qquad \text{Fp.} \int_0^{1/2} \frac{dt}{t} = -\log 2, \tag{II.2.25}$$

where one passes from one to the other by a C^∞ homeomorphism $x = 2t, t = x/2$.

3. F is no longer the analytic extension of $F(\lambda)$ up to $\lambda = 0$.
One sees immediately that $F(\lambda)$ goes to ∞unless λ tends to 0; the finite part Fp. $\int_a^b g(x)\,dx$ is the limit of $F(\lambda) - \frac{A_1}{\lambda}$ unless λ goes to 0.

It would seem that these difficulties do not appear if one of the exponents λ_ν, say λ_1, is equal to 1; but if, without any other exponent equal to 1, one of them is an integer, so when one considers $g(x)\varphi(x)$ for $\varphi \in \mathcal{D}$, one of the exponents will be equal to 1.

II.2.4. *Monomial pseudofunctions.* For $m \in \mathbb{C}$, call Pf.$(x^m)_{x>0}$ the pseudofunction distribution defined by

$$\begin{aligned} \text{Pf.}(x^m)_{x>0} \cdot \varphi &= \text{Fp.} \int_0^\infty x^m \varphi(x)\,dx \\ &= \lim_{\varepsilon \to 0} \left[\int_\varepsilon^\infty x^m \varphi(x)\,dx + \varphi(0)\tfrac{\varepsilon^{m+1}}{m+1} + \varphi'(0)\tfrac{\varepsilon^{m+2}}{m+2} + \cdots + \tfrac{\varphi^{(k)}(0)}{k!}\tfrac{\varepsilon^{m+k+1}}{m+k+1} \right]. \end{aligned} \tag{II.2.26}$$

The symbol Pf. is useless if $\text{Re}(m) > -1$, and the number of terms taken in the square bracket depend on the value of m; if m is a negative integer, the term with $\frac{\varepsilon^0}{0}$ must be replaced by $\log \varepsilon$.

$[\text{Pf.}(x^m)_{x>0}] \cdot \varphi$ is an analytic function of the complex variable m, except for when m is a negative integer. If m is not an integer ≤ 0, one clearly has

$$\frac{d}{dx}\left[\text{Pf.}(x^m)_{x>0}\right] = \text{Pf.}m(x^{m-1})_{x>0}. \tag{II.2.27}$$

In other words, the derivative of this pseudofunction is obtained by the ordinary rule for differentiating a monomial. In effect, the formula is exact for $\text{Re}(m) > 0$; so if one calls S and T the distributions which appear in the two equalities below, $S(\varphi)$ and $T(\varphi)$ are equal for $\text{Re}(m) > 0$, and they are analytic functions of the complex variable m, so they are identical. But if m is an integer ≤ 0, $m = -l$, this method of analytic continuation is no longer valid and one can show by direct calculation that

$$\begin{cases} \frac{d}{dx}\left[\text{Pf.}\left(\frac{1}{x^l}\right)_{x>0}\right] = \text{Pf.}\left(\frac{-l}{x^{l+1}}\right)_{x>0} + (-1)^l \frac{\delta^{(l)}}{l!} \\ \frac{d}{dx}\left[\text{Pf.}\left(\frac{1}{x^l}\right)_{x<0}\right] = \text{Pf.}\left(\frac{-l}{x^{l+1}}\right)_{x<0} - (-1)^l \frac{\delta^{(l)}}{l!} \end{cases} \tag{II.2.28}$$

The pseudofunction distribution Pf.$\frac{1}{x}$, which is the derivative of $\log|x|$, can also be written pv.$\frac{1}{x}$ because one has

$$\text{Pf.}\frac{1}{x}\cdot\varphi = \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{-\varepsilon}\frac{\varphi(x)}{x}dx + \int_{\varepsilon}^{\infty}\frac{\varphi(x)}{x}dx\right] = \text{pv.}\int_{-\infty}^{\infty}\frac{\varphi(x)}{x}dx, \tag{II.2.29}$$

where pv. denotes the Cauchy principal value, whose existence is guaranteed by the differentiability of φ.

Combining the two equalities in (II.2.28) one obtains

$$\frac{d}{dx}\text{Pf.}\left(\frac{1}{x^l}\right) = \text{Pf.}\left(\frac{-l}{x^{l+1}}\right). \tag{II.2.30}$$

Generally one has to consider the family of pseudofunctions

$$\begin{cases} Y_m = \frac{1}{\Gamma(m)}\text{Pf.}(x^{m-1})_{x>0}, & m \text{ not an integer} \leq 0 \\ Y_{-l} = \delta^{(l)}, & m = -l, \text{ an integer} \leq 0. \end{cases} \tag{II.2.31}$$

Regarding the definition of monomial pseudofunctions, one can see that if m tends toward an integer ≤ 0, $Y_m(\varphi)$ remains a continuous function of m but for the factor $\frac{1}{\Gamma(m)}$. $Y_m(\varphi)$ is thus an entire function. On the other hand, one always has the differentiation formula

$$\frac{d}{dx}Y_m = Y_{m-1}.$$

These remarks form the basis of the theory of fractional differentiation and integration.

VI. Convolution Products

VI.5. **Convolution products in the case of noncompact support.**

VI.5.1. *Application: fractional differentiation.*

We take as a particular distribution from $\mathcal{D}'_+$ the distribution Y_m defined by the formula (II.2.31). For fixed $\varphi \in \mathcal{D}_-$, $Y_m(\varphi)$ is an entire function; one can also say that Y_m is an entire function with which takes values in $\mathcal{D}'_+$.

One has the following convolution formula:

$$Y_p * Y_q = Y_{p+q}. \tag{VI.5.6}$$

In effect this formula is clear when p and q are complex numbers with positive real part, because then the symbol Pf. is unnecessary, Y_p and Y_q are functions, and the previous formula may be rewritten:

$$\int_0^x \frac{(x-t)^{p-1}t^{q-1}}{\Gamma(p)\Gamma(q)}dt = \frac{x^{p+q-1}}{\Gamma(p+q)} \quad \text{for } x>0, \tag{VI.5.7}$$

which is a classic property of Euler functions. So the two functions of p and q, $Y_p * Y_q$ and Y_{p+q}, which are equal when $\mathrm{Re}(p)>0$ and $\mathrm{Re}(q)>0$, are also equal when p and q**???**

The formula (VI.5.6) allows us to write, for any m, Y_m in the form Y^{*m} since for $m \in \mathbb{Z}$, it's the m^{th} power of Y in the algebra $\mathcal{D}'_+$. We can thus define the primitive and derivative of a complex order m of T by the formulas

$$\int_0^x I^m T = Y_m * T; \qquad D^m T = Y_{-m} * T. \tag{VI.5.8}$$

One has the following formulas as consequences of (VI.5.6)

$$\begin{aligned} I^p(I^q T) &= I^{p+q}(T); \\ D^p(D^q T) &= D^{p+q}(T); \\ D^m(I^m T) &= I^m(D^m T) = T. \end{aligned} \tag{VI.5.9}$$

One recovers the classical formulas (which usually rely on restrictive hypotheses of differentiability; see (VI.3.13)], but are now valid without restriction. For every distribution which has support bounded to the left, $I^m T$ depends continuously on the distributional integration of T and analytically of the complex order of integration m.

For an integer $m>0$, $D^m S$ is just the ordinary derivative, by the formula (II.2.31). $I^m S$ can be any primitive of S: this is only when the support is bounded at left; everything is different for a polynomial of degree $\le m-1$, which necessarily has unbounded support to the left (see also formula (VI.5.24)).

One can see from these formulas that differentiation and integration of distributions in $\mathcal{D}'_+$ are operations with the same nature: they are convolution operations.

If one would like to define the same notions for distributions with support bounded to the right, it is necessary to consider the distributions $(\check{Y})_m = (Y_m)\check{}$ and define the m^{th} power of the operators $(-I)$ and $(-D)$ by

$$(-I)^m S = \check{Y}_m * S, \qquad (-D)^m S = \check{Y}_{-m} * S. \tag{VI.5.10}$$

These two operations can be defined at once if S has support bounded to the left and right, that is to say, has compact support. This results, in particular, in making $m=1$, if S is a distribution with compact support, its only primitive with support bounded to the left is $Y * S$, and its only primitive with support bounded to the right is $-\check{Y} * S$.

This will give us a new procedure for finding the primitive of a distribution for any T (see II, §4). We shall choose any function α, infinitely differential in the usual sense, equal to 0 for $x \leq -c$ and equal to 1 for $x \geq c > 0$; and we put

$$S = \alpha S + (1-\alpha)S; \tag{VI.5.11}$$

αS has support bounded to the left and $(1-\alpha)S$ has support bounded to the right. One particular primitive of S will thus be

$$(Y * \alpha S) + \left[-\check{Y} * (1-\alpha)S\right]. \tag{VI.5.12}$$

This procedure is also understood in the literature as a solution to the equation

$$\tfrac{\partial}{\partial c}T = S;$$

one replaces Y with the linear measure $Y_{x_1} \times \delta_{x_2 x_3 \dots x_n} = Y_1$ (by (VI.5.9)), and one uses the same function $\alpha(x_1)$.

If we put

$$_aY_m = [e^{ax}]Y_m, \tag{VI.5.13}$$

the formula (IV.4.14) and (VI.5.6) show that one also has

$$_aY_p * {}_aY_q = {}_aY_{p+q} \quad \text{whence } {}_aY_m = ({}_aY)^{*m}. \tag{VI.5.14}$$

So for some distribution $T \in \mathcal{D}'_+$ and complex number m, we put

$$\begin{cases} {}_aI^m T = {}_aY_m * T \\ {}_aD^m T = {}_aY_{-m} * T; \end{cases} \tag{VI.5.15}$$

the operators $_aI^m$ and $_aD^m$ are again integration and differentiation operators of complex order. But one sees immediately that

$$_aY_{-l} = [e^{ax}]\delta' = \delta' - a\delta, \tag{VI.5.16}$$

so the operation $_aD$ is differentiation composed with

$$_aDT = \frac{dT}{dx} - aT \tag{VI.5.17}$$

and $_aD^m$ is the m^{th} power of the operation $_aD$. In particular, if m is an integer ≥ 0, $T = {}_aI^m S$ is the unique solution with bounded support to the left of the differential equation of degree m

$$\left(\frac{d}{dx} - a\right)^m T = S. \tag{VI.5.18}$$

This procedure indicates that the formulas (VI.5.11) and (VI.5.12) will give a solution that depends continuously on T, no matter what kind of support T may have.

Besides, this procedure may be understood for differential equations with nonconstant coefficients, as a generalization of the convolution products of Volterra. On the other hand, using direct products in $\mathbb{R}^n$, one may generalize the formulas of IV, §5, ex. 3, with derivatives of noninteger order in relation to the several variables $x_1, x_2, \dots, x_n$. It would also be easy to multiply the applications of this paragraph to the symbolic calculus of differential equations, to simplify (or render correct) the formulas of current usage, and to find new ones.

$$\begin{cases} \nabla^m T = S \rightrightarrows J^m \nabla^m T = T = J^m S \\ J^m S = T \rightrightarrows \nabla^m J^m S = S = \nabla^m T. \end{cases} \tag{VI.5.24}$$

$$E(x) S * T = [E(x) S] * [E(x) T] \tag{IV.4.14}$$

VII. Fourier Transforms

Summary: It is in the domain of the series and especially of the Fourier integral that the necessity of setting out the framework of functions is the most pressing.

§1 lays out the Fourier series. All periodic distributions have a Fourier series, which converges to this distribution; all trigonometric series whose coefficients are of slow growth converge to a periodic distribution of which it is the Fourier series. This theorem suffices for most application techniques.

§2 recalls the usual classical properties of the Fourier transform. The reciprocity formula (VII.2.3) and Parseval's formulas (VII.2.4) are the only necessary ones for understanding the sequel. It will not be possible to define the Fourier transformation for all distributions, but only for those of a subspace $\mathcal{S}'$ of $\mathcal{D}'$, the tempered distributions. $\mathcal{S}'$ is the dual of a space $\mathcal{S}$ of infinitely differentiable functions which is strictly larger than the space $\mathcal{D}$ of Chapter 1.

VII.1. **Fourier Series.**

VII.1.1. *Distributions on the torus.*

We represent the torus $\mathbb{T}^n$ as the quotient group of $\mathbb{R}^n$ by its subgroup $\mathbb{Z}^n$, the n^{th} cross-product of the group $\mathbb{Z}$ of integers. A point $x \in \mathbb{T}^n$ is represented by the n coordinates $x_1, x_2, \ldots, x_n$ which are real numbers modulo 1. $\mathbb{T}^n$ is a C^∞ compact manifold on which there exists a Haar measure $dx = dx_1 \cdot dx_2 \ldots dx_n$ and the partial derivatives $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}$ (see Chapter I, §5). The properties demonstrated for $\mathbb{R}^n$ in the preceding chapters are valid on $\mathbb{T}^n$ when they are of local character. Global considerations introduce, in contrast, all the questions of integration (Chapter II), problems with algebraic topology (for example, when $n = 1$ a distribution T does not possess a primitive if its integral is null). The theorems XXI, XXII, XXIII, of Chapter III are true on small enough open sets, and remain true on $\mathbb{T}^n$ if one replaces a unique derivative with a finite sum of derivatives (Theorems XXII and XXIII of Chapter VI). The convolution product has the same properties as on $\mathbb{R}^n$, simplified by the compactness of $\mathbb{T}^n$.

To distinguish the torus $\mathbb{T}^n$ from the space $\mathbb{R}^n$, we will call $f(x)$, T a function or a distribution on the torus, the scalar product will be denoted by $T \cdot \varphi$, and the convolution product by $S * T$.

VII.1.2. *Fourier Series.*

Theorem VII.1. *To any distribution T on the torus, one may give Fourier coefficients, defined by*

$$\begin{cases} a_l(T) = T \cdot e^{-2\pi i l \cdot x} \\ l = \{l_1, l_2, \ldots, l_n\}, \end{cases} \tag{VII.1.1}$$

where the l_j are integers; $l \cdot x = l_1x_1 + l_2x_2 + \cdots + l_nx_n$. These coefficients are of slow growth for $|l| \to \infty$:

$$\begin{cases} \lim_{|l|\to\infty} \frac{|a_l|}{(1+|l|^2)^k} & \textit{for suff large } k \\ |l| = \sqrt{l_1^2 + l_2^2 + \cdots + l_n^2}. \end{cases} \tag{VII.1.2}$$

The Fourier series formed of these coefficients converges in $\mathcal{D}'(\mathbb{T}^n)$ to T:

$$\sum a_l(T)e^{2\pi il\dot{x}} = T. \tag{VII.1.3}$$

Conversely, if $\{b_l\}$ is an arbitrary sequence of complex numbers which is of slow growth as $|l| \to \infty$, the trigonometric series $\sum b_l e^{2\pi il\cdot x}$ converges in $\mathcal{D}'$ to a distribution T which has the b_l as Fourier coefficients. A trigonometric series cannot converge unless it has coefficients of slow growth.

This theorem suppresses the usual awkward distinction of Fourier series and trigonometric series which are not Fourier series. Additionally, it renders useless the "summation procedures" for showing the convergence of a Fourier series.

(1) The formula (VII.1.1) immediately gives

$$\begin{aligned} a_l(S * T) &= (S_\xi \otimes T_\eta) exp^{-2\pi il\cdot(\xi+\eta)} \\ &= a_l(S)a_l(T) \end{aligned} \tag{VII.1.4}$$

Besides,

$$a_l(\delta) = 1; \qquad a_l\left(\frac{\partial\delta}{\partial x_k}\right) = 2\pi il_k, \tag{VII.1.5}$$

whence

$$a_l\left(\frac{\partial T}{\partial x_k}\right) = \left(\frac{\partial\delta}{\partial x_k}\right) a_l(T) = 2\pi il_k a_l(T). \tag{VII.1.6}$$

As the distribution T is a finite sum of derivatives of continuous functions, and the Fourier coefficients of a continuous function are limited, the coefficients $a_l(T)$ are of slow growth as $|l| \to \infty$, whatever T may be.

The Fourier series of the Dirac δ converges to δ in $\mathcal{D}'(\mathbb{T}^n)$; in effect

$$\sum_l e^{2\pi il\cdot x} \cdot \varphi = \sum_l a_{-l}(\varphi) = \varphi(0) = \delta \cdot \varphi \tag{VII.1.7}$$

(because φ is C^∞, so its Fourier series is convergent, in particular, for $x = 0$). The continuity of the convolution product also shows that the Fourier series of T converges to T, for any T, because

$$\begin{aligned} \sum_l a_l(T)e^{2\pi il\cdot x} &= \sum_l T_\xi \cdot e^{2\pi i\cdot(x-\xi)} \\ &= \sum_l T * e^{2\pi il\cdot x} \\ &= T * \delta = T. \end{aligned} \tag{VII.1.8}$$

(2) Now if b_l is a sequence of complex numbers of slow growth, the numerical series $\sum_l |b_l|/\left(1+|l|^2\right)^k$ is convergent for large enough k, thus the series

$$\sum_l \frac{|b_l|}{\left(1+|l|^2\right)^k} e^{2\pi i l\cdot x}$$

converges uniformly to a continuous function $f(x)$; the derivative of the series

$$\sum_l b_l e^{2\pi i l\cdot x}$$

thus also converges to the distribution $T = \left(1-\frac{\Delta}{4\pi^2}\right)^k f$. This last distribution T admits the b_l as Fourier coefficients. More generally, if a series $\sum b_l e^{2\pi i l\cdot x}$ converges to a distribution T, it will admit the b_l as Fourier coefficients (which obliges the b_l to be of slow growth as $|l| \to \infty$). Because of the calculation of $a_l(T)$ by the method of orthogonality of exponents, following the classical term-by-term method, $a_l(T)$ reduces to b_l.

Remarks

(1) This demonstration shows us at the same time that every distribution T on the torus is the $\left(1-\frac{\Delta}{4\pi^2}\right)^k$ of a continuous function for large enough k.

(2) If a series $\sum_l b_l e^{2\pi i l\cdot x}$ converges on an open domain Ω in $\mathbb{T}^n$, Theorem XXIII of Chapter VI and formula (VI.6.22), applied in Ω to functions $b_l e^{2\pi i l\cdot x}$ which converge to 0 in $\mathcal{D}'(\mathbb{T}^n)$, when $|l| \to \infty$; the series converges not only on Ω but on $\mathbb{T}^n$, and it is a Fourier series.

(3) We will see that if $c(l)$ is a series of slow growth (a polynomial in l, for example), the functions $c(l)e^{2\pi i l\cdot x}$ converge to 0 in $\mathcal{D}'(\mathbb{T}^n)$ as $|l| \to \infty$. A fortiori, one may say that "the functions $e^{2\pi i l\cdot x}$ converge to 0 in $\mathcal{D}'(\mathbb{T}^n)$ as $|l| \to \infty$, faster than any power of $|l|^{-1}$". But this language risks ambiguity and error. In effect, no matter what the sequence $c(l)$ may be, faster growth than any power of $|l|$ as $|l| \to \infty$, the functions $c(l)e^{2\pi i l\cdot x}$ are not bounded in $\mathcal{D}'(\mathbb{T}^n)$. Same properties for the functions $c(h)e^{2\pi i h\cdot x}$ in $\mathcal{D}'(\mathbb{R}^n)$ or similarly $\mathcal{B}'(\mathbb{R}^n), h \in \mathbb{R}^n$.

(4) If $\varphi \in \mathcal{D}(\mathbb{T}^n)$, the $a_l(\varphi)$ form a sequence of rapid decay for $|l| \to \infty$:

$$\lim_{|l|\to\infty} |l|^k |a_l| = 0 \quad \text{for any } k.$$

In this way, the Fourier series puts $\mathcal{D}(\mathbb{T}^n)$ and $\mathcal{D}'(\mathbb{T}^n)$ into correspondence (by topological vector space isomorphism) with the spaces of numerical sequences of rapid decay and the space of numerical sequences of slow growth, respectively. The duality between $\mathcal{D}(\mathbb{T}^n)$ and $\mathcal{D}'(\mathbb{T}^n)$ corresponds to the duality of the two sequence spaces by Parseval's formula

$$T \cdot \varphi = \sum_l a_l(T) a_{-l}(\varphi). \tag{VII.1.9}$$

VII.1.3. *Fourier series of elliptic functions.*

Let $\mathbf{p}(z)$ be a standard elliptic function corresponding to the periods 1 and i. One may consider it as a meromorphic function on the 2-dimensional torus. It does not define a distribution, as it is not summable in a neighborhood of the origin, but after having seen Chapter II, §3, example 3, pv.$\mathbf{p}(z)$ is a distribution of which we may seek the Fourier series. One has, according to formula (II.3.27):

$$\frac{\partial}{\partial \bar{z}}\left[\mathrm{pv}.\mathbf{p}(z)\right] = -\pi \frac{\partial \delta}{\partial z}. \tag{VII.1.10}$$

The development of the Fourier series following the $e^{2\pi i(px+qy)}$ (p, q are integers) shows that this partial differential equation has only one solution, up to an additive constant, because one has

$$\frac{1}{2} 2\pi i(p+iq) a_{p,q}\left[\mathrm{pv}.\mathbf{p}(z)\right] = -\pi \frac{1}{2} 2\pi i(p-iq);$$

whence

$$\mathrm{pv}.\mathbf{p}(z) = a_{0,0} - \pi \sum_{(p,q)\neq(0,0)} \frac{p-iq}{p+iq} e^{2\pi i(px+qy)}. \tag{VII.1.11}$$

The coefficients, all modulo 1, are bounded and hence of slow growth, and the series converges. The same procedure shows that there exists, up to an additive constant, an elliptic function of only periods $(1, i)$, having poles given in polar form in a neighborhood of each of them, provided that the division by $p + iq$ is possible; this exists except when the coefficient $a_{0,0}$ of the 2nd term of the equation analogous to (VII.1.10) is 0, i.e., the sum of the residues is 0. This elliptic function is determined by the Fourier series of the associated pseudofunction pv..

VII.1.4. *Finite difference equations.*

Suppose there is a doubly-indexed sequence $\{a_{p,q}\}_{p,q\in\mathbb{Z}}$ of slow growth for $p^2+q^2 \to \infty$, satisfying the following finite difference equation:

$$a_{p+1,q} + a_{p-1,q} + a_{p,q+1} + a_{p,q-1} - 4\lambda a_{p,q} = b_{p,q}, \tag{VII.1.12}$$

where the $b_{p,q}$ are given, and assumed to be of slow growth for $p^2 + q^2 \to \infty$. If one considers the $a_{p,q}$ and the $b_{p,q}$ as the Fourier coefficients of distributions A and B on the torus, one will have to have

$$2(\cos 2\pi x + 2\pi y - 2\lambda)A = B. \tag{VII.1.13}$$

(a) If λ isn't real, or if it is real but $|\lambda| > 1$, the division is immediate and the only possible $a_{p,q}$, which will be of slow growth for $p^2 + q^2 \to \infty$, are the Fourier coefficients of

$$A = \frac{B}{2(\cos 2\pi x + \cos 2\pi y - 2\lambda)}.$$

(b) If λ is real, $|\lambda| < 1$, $\lambda \neq 0$, the collection V defined by

$$\cos 2\pi x + \cos 2\pi y - 2\lambda = 0$$

does not have multiple points, division is possible by Theorem VIII of Chapter V; but there exist infinitely many solutions for A; the difference between any two of them is a simple arbitrary layer borne by the collection V.

(c) If $\lambda = 0$, the collection V of points having two distinct tangents, we know how to resolve problems with division. If $\lambda = \pm 1$, V is reduced to one of the points $(0,0)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$, two isolated points, the division is not carried out as in the case of Chapter V, but more easily.

VII.1.5. *Distributions on the torus and periodic distributions on* $\mathbb{R}^n$.

One sees immediately, upon defining distributions in a neighborhood of each point and upon "gathering the morsels" (Theorem IV of Chapter I), that one may establish a bijective correspondence between the distributions on the torus and the periodic distributions on $\mathbb{R}^n$, of period 1 for all coordinates, i.e., satisfying

$$\tau_l T = T \tag{VII.1.14}$$

for any $l = (l_1, l_2, \ldots, l_n), l_j \in \mathbb{Z}$.

For convenience, we will simply call such a distribution *periodic*; we denote the associated distribution on the torus by $\dot{T}$.

These two distributions are locally identical, thus possess the same local regularity characteristics (differentiability, order, etc.). All periodic distributions are thus the $\left(1 - \frac{\Delta}{4\pi^2}\right)^k$ of a continuous periodic function, for large enough k.

A periodic distribution is almost-periodic (Chapter VI, §9).

One sees immediately that under these conditions, the products $\dot{S} * \dot{T}$, $\dot{T} \cdot \dot{\varphi}$, are the products $S * T$, $T \cdot \varphi$ defined on $\mathbb{R}^n$ for the distributions p p. (Formulas VI.9.3, VI.9.4). Similarly, the Fourier coefficients $a_l(\dot{T})$ are those of the distribution p p. T (formula VI.9.7), the coefficients a_λ corresponding to $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ (whose noninteger coordinates are 0).

Now let T be a summable distribution on $\mathbb{R}^n$ ($T \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$. One may define the direct image of the torus, by the canonical projection $x \mapsto x$ from $\mathbb{R}^n$ to $\mathbb{T}^n$ (this canonical projection is not C^∞, whence the restriction regarding the summability of T). This image is identified with a periodic distribution on $\mathbb{R}^n$, which we will call ϖT, the periodic transform of T.

Let $\varphi \in \mathcal{D}_{L^1}(\mathbb{R}^n)$; $\dot{\psi} \in \mathcal{D}(\mathbb{T}^n)$; $S, S_1, S_2 \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$; $\dot{T} \in \mathcal{D}'(\mathbb{T}^n)$; and one has

$$\begin{cases} \varpi & = \sum_l \varphi(x - l) \\ \varpi S & = \sum_l \tau_l S \\ (\varpi S)^{\cdot} \cdot \dot{\psi} & = S \cdot \psi \\ \dot{T} \cdot (\varpi\varphi)^{\cdot} & = T \cdot \varphi \\ (\varpi S)^{\cdot} \cdot (\varpi\varphi)^{\cdot} & = S \cdot \varpi\psi = \varpi S \cdot \varphi \end{cases} \tag{VII.1.15}$$

$$\begin{cases} (\varpi S_1)^{\cdot} * (\varpi S_2)^{\cdot} & = (S_1 * \varpi S_2)^{\cdot} = (\varpi S_1 * S_2)^{\cdot} = (\varpi(S_1 * S_2))^{\cdot} \\ (\varpi S)^{\cdot} * \dot{T} & = (S * T)^{\cdot} \end{cases} \tag{VII.1.16}$$

These formulas allow calculation of the Fourier coefficients of a periodic distribution T on $\mathbb{R}^n$, without passing to the scalar product on the torus.

$$a_l(T) = T \cdot e_l, \tag{VII.1.17}$$

e_l being an arbitrary function in $\mathcal{D}$ (or $\mathcal{D}_{L^1}$) whose periodic transform ϖe_l is $e^{-2\pi i l \cdot x}$. Usually, one now takes if T is a function f, e_l to be equal to $e^{-2\pi i l \cdot x}$ on a cube of its periods and besides, this may express $a_l(f)$ by an integral taken over a cube of these periods; this method is unusable if T is a distribution because the chosen function e_l is discontinuous.

But one also has

$$a_l(T) e^{2\pi i l \cdot x} = T * e^{2\pi i l \cdot x} = T * E_l, \tag{VII.1.18}$$

E_l being any distribution in $\mathcal{D}'_{L^1}$ for which the periodic transform ϖE_l is $e^{2\pi i l \cdot x}$. So one may take for E_l a standard discontinuous function supported on a cube of periods in $\mathbb{R}^n$.