The Laplace Equation ERIN PEARSE

I. Introduction

Definition 1. Among the most important and ubiquitous of all partial differential equations is *Laplace's Equation*:

$$\Delta u = 0,$$

where the *Laplacian operator* Δ acts on the function $u : U \to \mathbb{R}$ (U is open in $\mathbb{R}^n$) by taking the sum of the unmixed partial derivatives. For example:

$n = 1$: $\Delta u = \frac{\partial^2 u}{\partial x^2} = u'' = 0$
In this simple case, the solution $u = ax + b$ is found by integrating twice.

$n = 2$: $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0$
Here $u = u(x_1, x_2)$ and the solution is already much more difficult to obtain.

$\vdots$

$n = N$: $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_N^2} = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2} = 0$

A solution to Laplace's equation is a function u satisfying $\Delta u = 0$.

Definition 2. When combined with boundary conditions such as

$$u = g \text{ on } \partial U,$$

finding a solution to Laplace's equation is referred to as the *Dirichlet Problem.*

Definition 3. In general, a C^2 function u satisfying $\Delta u = 0$ is called a *harmonic function.*

Definition 4. The closely related nonhomogeneous case is known as *Poisson's Equation*:

$$-\Delta u = f.^1$$

II. Physical Interpretations

Laplace's equation in 2 and 3 dimensions occurs in time-independent problems involving potentials (e.g. electrostatic, gravitational) and velocity in fluid mechanics. A solution to Laplace's equation can also be interpreted as a steady-state temperature distribution. In a typical interpretation, u denotes the density or concentration of some quantity in equilibrium:

u **denotes chemical concentration:**
Fick's law of diffusion: $\Delta u = 0$

u **denotes temperature:**
Fourier's law of heat conduction: $\Delta u = 0$

u **denotes electrostatic potential:**
Ohm's law of electrical conduction: $\Delta u = 0$

[1]The (-) here is for consistency with notation for higher-order elliptic operators.

III. The Fundamental Solution to Laplace's Equation

The basic idea for deriving the fundamental solution is to exploit symmetry by observing that Laplace's equation is rotation invariant:

Lemma 1. Suppose $\Delta u = 0$. If A is an orthogonal $n \times n$ matrix and we define $v(x) := u(Ax)$ for $(x \in \mathbb{R}^n)$, then $\Delta v = 0$.

Proof. Using summation notation, write $A = a_{ij}$ and consider $u(y)$ where $y_i = a_{ij}x_j$. Then

$$\begin{aligned}\frac{\partial}{\partial x_k} &= \frac{\partial}{\partial y_i}\left(\frac{\partial y_i}{\partial x_k}\right)\\ &= \frac{\partial}{\partial y_i} a_{ik}\\ &= a_{ik}\frac{\partial}{\partial y_i},\end{aligned}$$

so taking partials again gives

$$\begin{aligned}\frac{\partial^2}{\partial x_k^2} &= a_{ik}\frac{\partial}{\partial y_i}\left[\frac{\partial}{\partial x_k}\right]\\ &= a_{ik}\frac{\partial}{\partial y_i}\left[\frac{\partial}{\partial y_j}a_{jk}\right]\\ &= a_{ik}a_{jk}\frac{\partial^2}{\partial y_i \partial y_j}\\ &= \frac{\partial^2}{\partial y_k^2}\end{aligned}$$

□

Now that we have established the radial symmetry of Laplace's equation, we know to look for solutions which are functions of $r = |x|$, i.e., of the form

$$u(x) = v(r),$$

where $r = |x|$ and v is to be chosen so that $\Delta u = 0$ holds. First, note that we have

$$\begin{aligned} r = |x| &= \left(x_1^2 + x_2^2 + \ldots + x_n^2\right)^{1/2}\\ \implies \frac{\partial r}{\partial x_i} &= \frac{1}{2}\left(x_1^2 + x_2^2 + \ldots + x_n^2\right)^{-1/2}(2x_i) = \frac{x_i}{r}\end{aligned}$$

for $i = 1, 2, \ldots, n$ and $x \neq 0$. So then from $u(x) = v(r)$ we have

$$u_{x_i} = v'(r)\tfrac{x_i}{r}$$

and using the product rule, we also have

$$\begin{aligned}u_{x_i x_i} &= v''(r)\left(\tfrac{x_i}{r}\right)^2 + v'(r)\left(\frac{x_i}{\left(x_1^2+\ldots+x_n^2\right)^{1/2}}\right)'\\ &= v''(r)\left(\tfrac{x_i}{r}\right)^2 + v'(r)\left(\frac{r - x_i^2/r}{r^2}\right)\\ &= v''(r)\left(\tfrac{x_i}{r}\right)^2 + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)\end{aligned}$$

so summing over i gives

$$\begin{aligned}
\Delta u &= \sum_{i=1}^{n} \left(v''(r)\tfrac{x_i^2}{r^2} + v'(r)\left(\tfrac{1}{r} - \tfrac{x_i^2}{r^3}\right)\right) \\
&= v''(r)\sum_{i=1}^{n} \tfrac{x_i^2}{r^2} + v'(r)\sum_{i=1}^{n}\left(\tfrac{1}{r} - \tfrac{x_i^2}{r^3}\right) \\
&= v''(r)\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{r^2} + v'(r)\left(\frac{n}{r} - \frac{(x_1^2 + x_2^2 + \ldots + x_n^2)}{r^3}\right) \\
&= v''(r)\frac{r^2}{r^2} + v'(r)\left(\frac{n}{r} - \frac{r^2}{r^3}\right) \\
&= v''(r) + v'(r)\left(\tfrac{n-1}{r}\right)
\end{aligned}$$

Hence, $\Delta u = 0 \iff v''(r) + v'(r)\frac{n-1}{r} = 0$.
For $v' \neq 0$, this gives $v''(r) = v'(r)\frac{1-n}{r} \implies \frac{v''}{v'} = \frac{1-n}{r}$. Now note that

$$\tfrac{v''}{v'} = (\log|v'|)'.$$

Then

$$\begin{aligned}
(\log|v'|)' = \tfrac{1-n}{r} \qquad \implies \qquad \log|v'| &= (1-n)\log r + c \\
|v'| &= e^{(1-n)\log r}e^c \\
v' &= ae^{\log r^{(1-n)}} \\
v' &= ar^{(1-n)} \\
v' &= \tfrac{a}{r^{(n-1)}}
\end{aligned}$$

Hence

$$n = 2 \implies v' = \tfrac{a}{r} \implies v(r) = b\log r + c \text{ for } r > 0.$$

Note that the assumption $(r > 0)$ is justified because r denotes radius. Similarly,

$$n \geqslant 3 \implies v' = \tfrac{a}{r^{(n-1)}} \implies v(r) = \tfrac{b}{r^{(n-2)}} + c.$$

Thus we have obtained

$$v(r) = \begin{cases} b\log r & n = 2 \\ \frac{b}{r^{n-2}} & n \geqslant 3 \end{cases}$$

for some constants b, c.

At this point, we introduce some convenient "shorthand" notation:

$\alpha(n) = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ vol of unit ball in $\mathbb{R}^n$
$\alpha(n)r^n =$ vol of r-ball in $\mathbb{R}^n$
$n\alpha(n) =$ area of unit sphere in $\mathbb{R}^n$
$n\alpha(n)r^{n-1} =$ area of r-sphere in $\mathbb{R}^n$

Definition 5. Now to normalize on the unit ball (and for the sake of tidy statements of later results) we define

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & n = 2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & n \geqslant 3 \end{cases}$$

for $x \in \mathbb{R}^n, x \neq 0$ to be the *fundamental solution of Laplace's equation.*

IV. Solving Poisson's Equation

Now we generalize and extend our solution for $\Delta u = 0$ to solve Poisson's equation $-\Delta u = f$.

Remark. Note that $\Phi(x)$ is harmonic by construction for $x \neq 0$, so $\Phi(x-y)$ is harmonic for $x \neq y$. Next, consider that for any $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \Phi(x-y)f(y)$ will also be harmonic for any $y \neq x$, as will any sum of such expressions. This would suggest the following reasoning:

$$\begin{aligned} \Delta u(x) = 0 & \\ &\implies \Delta_x \Phi\left(x-y\right) = 0 \\ &\implies \Delta_x \Phi\left(x-y\right) f\left(y\right) = 0 \\ &\implies \int_{\mathbb{R}^n} \Delta_x \Phi\left(x-y\right) f\left(y\right) dy = 0 \end{aligned}$$

so that we are essentially just slipping the Laplacian inside the integral to obtain as solution the convolution

$$u(x) = \int_{\mathbb{R}^n} \Phi\left(x-y\right) f\left(y\right) dy$$

Unfortunately, the singularity at $y = x$ prevents this simple approach and more care must be taken.

Theorem 1. Suppose that $f \in C_c^2(\mathbb{R}^n)$ and

$$u(x) = \int_{\mathbb{R}^n} \Phi\left(x-y\right) f\left(y\right) dy = \begin{cases} -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left(|x-y|\right) f\left(y\right) dy & n = 2 \\ \frac{1}{n(n-2)\alpha(n)}\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy & n \geqslant 3 \end{cases}.$$

Then u provides a solution of Poisson's equation, i.e.,

(i) $u \in C^2(\mathbb{R}^n)$, and

(ii) $-\Delta u = f$ in $\mathbb{R}^n$.

Proof. (i) We have

$$u\left(x\right) = \int_{\mathbb{R}^n} \Phi\left(x-y\right) f\left(y\right) dy = \int_{\mathbb{R}^n} \Phi\left(y\right) f\left(x-y\right) dy.$$

Thus, we make the substitution

$$\begin{aligned} u(x+he_i) - u(x) &= \int_{\mathbb{R}^n} \Phi(y) f(x+he_i-y)\,dy - \int_{\mathbb{R}^n} \Phi(y) f(x-y)\,dy \\ &= \int_{\mathbb{R}^n} \Phi(y) \left[f(x+he_i-y) - f(x-y) \right] dy \end{aligned}$$

to obtain the difference quotient

$$\tfrac{u(x+he_i)-u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \left[\tfrac{f(x+he_i-y)-f(x-y)}{h} \right] dy$$

so that we may calculate the partial derivative

$$\begin{aligned} \tfrac{\partial u}{\partial x_i}(x) &= \lim_{h\to 0} \tfrac{u(x+he_i)-u(x)}{h} \\ &= \lim_{h\to 0} \int_{\mathbb{R}^n} \Phi(y) \left[\tfrac{f(x+he_i-y)-f(x-y)}{h} \right] dy \\ &= \int_{\mathbb{R}^n} \Phi(y) \lim_{h\to 0} \left[\tfrac{f(x+he_i-y)-f(x-y)}{h} \right] dy \\ &= \int_{\mathbb{R}^n} \Phi(y) \tfrac{\partial f}{\partial x_i}(x-y)\,dy \end{aligned}$$

Note: we can justify moving the limit under the integral sign (third equality in the previous computation) because $f \in C_c^2(\mathbb{R}^n)$ gives us that f is uniformly continuous, and hence that this convergence is uniform.

Similarly, $f \in C_c^2(\mathbb{R}^n)$ gives us that $\frac{\partial u}{\partial x_i}$ is uniformly continuous and we can make a nearly identical repetition of this argument to obtain

$$\begin{aligned} \tfrac{\partial^2 u}{\partial x_j \partial x_i}(x) &= \lim_{h\to 0} \tfrac{1}{h} \left(\tfrac{\partial u}{\partial x_i}(x+he_j) - \tfrac{\partial u}{\partial x_i}(x) \right) \\ &= \lim_{h\to 0} \tfrac{1}{h} \int_{\mathbb{R}^n} \Phi(y) \left[\tfrac{\partial f}{\partial x_i}(x+he_j-y) - \tfrac{\partial f}{\partial x_i}(x-y) \right] dy \\ &= \int_{\mathbb{R}^n} \Phi(y) \lim_{h\to 0} \tfrac{1}{h} \left[\tfrac{\partial f}{\partial x_i}(x+he_j-y) - \tfrac{\partial f}{\partial x_i}(x-y) \right] dy \\ &= \int_{\mathbb{R}^n} \Phi(y) \tfrac{\partial^2 f}{\partial x_i \partial x_j}(x-y)\,dy \end{aligned}$$

Since $\int_{\mathbb{R}^n} \Phi(y) \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y)\,dy$ is continuous in x, $\frac{\partial^2 u}{\partial x_i^2} \implies u \in C^2(\mathbb{R}^n)$. □

Proof. (ii) By part (i) we have

$$\begin{aligned}
\Delta u(x) &= \sum_{i=1}^{n} \tfrac{\partial^2 u}{\partial x_i^2}(x) \\
&= \sum_{i=1}^{n} \int_{\mathbb{R}^n} \Phi(y) \tfrac{\partial^2 f}{\partial x_i^2}(x-y)\, dy \\
&= \int_{\mathbb{R}^n} \Phi(y) \left[\sum_{i=1}^{n} \tfrac{\partial^2 f}{\partial x_i^2}(x-y) \right] dy \\
&= \int_{\mathbb{R}^n} \Phi(y)\, \Delta_x f(x-y)\, dy
\end{aligned}$$

Now we need to isolate the singularity of Φ at 0, so we fix some $\varepsilon > 0$ and split the integral

$$\int_{\mathbb{R}^n} \Phi(y)\, \Delta_x f(x-y)\, dy = \underbrace{\int_{B(0,\varepsilon)} \Phi(y)\, \Delta_x f(x-y)\, dy}_{I_\varepsilon} + \underbrace{\int_{\mathbb{R}^n \sim B(0,\varepsilon)} \Phi(y)\, \Delta_x f(x-y)\, dy}_{J_\varepsilon}.$$

Now denoting $B := B(0,\varepsilon)$, we have

$$\begin{aligned}
|I_\varepsilon| &= \left| \int_B \Phi(y)\, \Delta_x f(x-y)\, dy \right| \\
&\leqslant \int_B |\Phi(y)\, \Delta_x f(x-y)|\, dy \\
&\leqslant \int_B |\Phi(y)| \cdot \|\Delta_x f(x-y)\|_\infty\, dy \\
&= \|\Delta_x f(x-y)\|_\infty \int_B |\Phi(y)|\, dy \\
&= C \left\| D^2 f \right\|_\infty \int_B |\Phi(y)|\, dy \\
&\leqslant \begin{cases} C\varepsilon^2 |\log \varepsilon| & n = 2 \\ C\varepsilon^2 & n \geqslant 3 \end{cases}.
\end{aligned}$$

Thus $\lim_{\varepsilon \to 0} |I_\varepsilon| = 0$. Turning to the other integral and denoting $\tilde{B} := \mathbb{R}^n \sim B(0,\varepsilon)$, $J_\varepsilon = \int_{\tilde{B}} \Phi(y)\, \Delta_x f(x-y)\, dy$, so we use integration by parts applying the formula

$$\int_U u_{x_i} v\, dx = -\int_U u v_{x_i} dx + \int_{\partial U} u v \nu^i dS$$

with $u_{x_i} = \Delta_x f(x-y)\, dy$ and $v = \Phi(y)$, so that

$$J_\varepsilon = \underbrace{-\int_{\tilde{B}} D\Phi(y) \cdot D_y f(x-y)\, dy}_{K_\varepsilon} + \underbrace{\int_{\partial B} \Phi(y) \tfrac{\partial f}{\partial \nu}(x-y)\, dS(y)}_{L_\varepsilon}$$

Now for L_ε, we see

$$\begin{aligned} |L_\varepsilon| &= \left| \int_{\partial B} \Phi(y) \tfrac{\partial f}{\partial \nu}(x-y)\, dS(y) \right| \\ &\leqslant \int_{\partial B} \left| \Phi(y) \tfrac{\partial f}{\partial \nu}(x-y) \right| dS(y) \\ &\leqslant \int_{\partial B} |\Phi(y)| \cdot \left\| \tfrac{\partial f}{\partial \nu}(x-y) \right\|_\infty dS(y) \\ &= \left\| \tfrac{\partial f}{\partial \nu}(x-y) \right\|_\infty \int_{\partial B} |\Phi(y)|\, dS(y) \\ &= \|Df\|_\infty \int_{\partial B} |\Phi(y)|\, dS(y) \\ &\leqslant \begin{cases} C\varepsilon\, |\log \varepsilon| & n=2 \\ C\varepsilon & n \geqslant 3 \end{cases} \end{aligned}$$

Thus $\lim\limits_{\varepsilon \to 0} |L_\varepsilon| = 0$.

Now for $K_\varepsilon = -\int_{\tilde{B}} D\Phi(y) \cdot D_y f(x-y)\, dy$, we use integration by parts again with $u_{x_i} = D_y f(x-y) dy$ and $v = D\Phi(y)$, to obtain

$$\begin{aligned} K_\varepsilon &= -\int_{\tilde{B}} \Delta\Phi(y) f(x-y)\, dy - \int_{\partial B} \tfrac{\partial \Phi}{\partial \nu}(y) f(x-y)\, dS(y) \\ &= -\int_{\partial B} \tfrac{\partial \Phi}{\partial \nu}(y) f(x-y)\, dS(y) \end{aligned}$$

where the first integral vanishes because we constructed Φ to be harmonic, in the first place (hence $\Delta\Phi = 0 \implies \int \Delta\Phi = 0$. More precisely, $\Phi(y)$ is harmonic for $y \neq 0$, but this integration takes place over the portion of $\mathbb{R}^n$ which is at least ε from 0, so we are okay.

Now we have by definition that $\frac{\partial \Phi}{\partial \nu}(y) := \nu \cdot D\Phi(y)$, so

$$\nu = \tfrac{-y}{|y|} \quad \text{and} \quad D\Phi(y) = \tfrac{-1}{n\alpha(n)} \tfrac{y}{|y|^n} \text{ for } y \neq 0,$$

which becomes

$$\nu = \tfrac{-y}{\varepsilon} \quad \text{and} \quad D\Phi(y) = \tfrac{-1}{n\alpha(n)} \tfrac{y}{\varepsilon^n}$$

on the sphere $\partial B(0, \varepsilon)$. Putting these together,

$$\begin{aligned} \tfrac{\partial \Phi}{\partial \nu}(y) = \nu \cdot D\Phi(y) &= \tfrac{-y}{\varepsilon} \cdot \left(\tfrac{-1}{n\alpha(n)} \tfrac{y}{\varepsilon^n} \right) \\ &= \tfrac{y \cdot y}{n\alpha(n)\varepsilon^{n+1}} \\ &= \tfrac{|y|^2}{n\alpha(n)\varepsilon^{n+1}} \\ &= \tfrac{\varepsilon^2}{n\alpha(n)\varepsilon^{n+1}} && (|y| = \varepsilon \text{ on } \partial B(0, \varepsilon)) \\ &= \tfrac{1}{n\alpha(n)\varepsilon^{n-1}} \end{aligned}$$

We substitute this expression into our formula for K_ε to obtain

$$K_\varepsilon = \tfrac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(x,\varepsilon)} f(y)\, dS(y).$$

Now we make the convenient definition

$$⨍_{\partial B(x,r)} f\, dS := \tfrac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} f\, dS$$

for the average value of f on the sphere $\partial B(x,r)$, so that we can rewrite this result as

$$K_\varepsilon = -⨍_{\partial B(x,\varepsilon)} f(y)\, dS(y).$$

Thus

$$\lim_{\varepsilon\to 0} |K_\varepsilon| = \lim_{\varepsilon\to 0} \left[-⨍_{\partial B(x,\varepsilon)} f(y)\, dS(y) \right] = -f(x).$$

Combining all terms,

$$\Delta u(x) = \lim_{\varepsilon\to 0} (I_\varepsilon + K_\varepsilon + L_\varepsilon) = 0 - f(x) + 0 = -f(x)$$

□

Remark. Φ is constructed to be harmonic away from the origin, i.e., $\Delta\Phi(x) = 0$ for $x \neq 0$, but what is $\Delta\Phi(0)$? Although this question is perhaps not quite well-formed, we can provide a fairly precise answer by defining $-\Delta\Phi(x) = \delta_0$ in $\mathbb{R}^n$, the Dirac delta "function" (distribution) in $\mathbb{R}^n$. Returning to our previous intuition, we can now make the formal computation

$$\begin{aligned}
\Delta u(x) &= \int_{\mathbb{R}^n} \Delta_x \Phi(x-y) f(y)\, dy \\
&= \int_{\mathbb{R}^n} -\delta_x f(y)\, dy \\
&= -\int_{\mathbb{R}^n} f(y)\, d\mu_x \\
&= -f(x)
\end{aligned}$$

where $d\mu_x$ is the measure defined by

$$d\mu_x(E) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

Remark. The previous theorem can be proven under less restrictive conditions. For an integrable function f on a domain Ω (and Φ as above), let u be defined on $\mathbb{R}^n$ by

$$u(x) = \int_\Omega \Phi(x-y) f(y) dy.$$

Then the previous result can be extended as follows: let f be bounded and locally Hölder continuous (with exponent $\alpha \leqslant 1$) in Ω. Then $u \in C^2(\Omega)$ and $\Delta u = f$ in Ω.

V. Mean-value Formulas

Theorem 2. If $u \in C^2(U)$ satisfies $\Delta u = 0$ for an open set U in $\mathbb{R}^n$, then $\forall B(x,r) \subset U$,

$$u(x) = \fint_{\partial B(x,r)} u\,dS = \fint_{B(x,r)} u\,dy.$$

Proof. For the first equality, define

$$\phi(r) := \fint_{\partial B(x,r)} u(y)\ dS(y) = \fint_{\partial B(0,1)} u(x+rz)\,dS(z)$$

so that

$$\begin{aligned}
\phi'(r) &= \fint_{\partial B(0,1)} Du(x+rz)\cdot z\ dS(z) \\
&= \fint_{\partial B(x,r)} Du(y)\cdot \tfrac{y-x}{r} dS(y) \\
&= \fint_{\partial B(x,r)} Du(y)\cdot \nu dS(y) \\
&= \fint_{\partial B(x,r)} \tfrac{\partial u}{\partial \nu}\ dS(y) \\
&= \tfrac{r}{n}\fint_{B(x,r)} \Delta u(y)dy \qquad \left(\int_{\partial U} \tfrac{\partial u}{\partial \nu} dS = \int_U \Delta u dx\right) \\
&= 0
\end{aligned}$$

Hence ϕ is constant, which implies

$$\phi(r) = \lim_{t\to 0}\phi(t) = \lim_{t\to 0}\fint_{\partial B(x,t)} u(y)\,dS(y) = u(x)$$

For the second equality, using polar coordinates gives

$$\begin{aligned}
\int_{B(x,r)} u\,dy &= \int_0^r \left(\int_{\partial B(x,s)} udS\right) ds \\
&= \int_0^r u(x)\,n\alpha(n)\,s^{n-1}ds \\
&= u(x)\,n\alpha(n)\int_0^r s^{n-1}ds \\
&= u(x)\,n\alpha(n)\,\tfrac{r^n}{n} \\
&= u(x)\,\alpha(n)\,r^n
\end{aligned}$$

So

$$\int_{B(x,r)} udy = u(x)\,\alpha(n)\,r^n \quad \Longrightarrow \quad u(x) = \tfrac{1}{\alpha(n)r^n}\int_{B(x,r)} udy = \fint_{B(x,r)} udy$$

□

Corollary. $u \in C^2(U)$ satisfies $u(x) = \fint_{\partial B(x,r)} u\, dS,\ \forall B(x,r) \subset U \quad \implies \quad u$ is harmonic.

Proof. If $\Delta u \not\equiv 0$, then $\exists B(x,r) \subset U$ such that $\Delta u > 0$ (or $\Delta u < 0$) on $B(x,r)$. But then for ϕ as above,

$$0 = \phi'(r) = \tfrac{r}{n} \int_{B(x,r)} \Delta u(y)\, dy > 0 \qquad \lightning$$

□

Theorem 3. $U \subset \mathbb{R}^n$ is open and bounded, $u \in C^2(U) \cap C(\bar{U})$ is harmonic within U.

(i) Suppose U is connected. Then $\exists x_0 \in U$ such that $u(x_0) = \max_{\bar{U}} \{u\} \implies u$ is constant on U.

(ii) $\max_{\bar{U}} \{u\} = \max_{\partial U} \{u\}$

Proof. (i) Suppose $\exists x_0 \in U$ such that $u(x_0) = M := \max_{\bar{U}} \{u\}$. Then $B(x_0, r) \subset U$ whenever $0 < r < \operatorname{dist}(\mathrm{x_0}, \partial \mathrm{U})$, so we can apply the mean-value theorem to get

$$M = u(x_0) = \fint_{B(x,r)} u dy.$$

Since M is the maximum of u on the ball, equality can only hold if $u \equiv M$ on the ball; i.e., it must be that $u(y) = M,\ \forall y \in B(x_0, r)$. This also shows $\{u = M\}$ is an open set:

$$u(x_0) = M \qquad \implies \qquad u(y) = M,\ \forall y \in B(x_0, r).$$

Moreover, $\{u = M\}$ is also relatively closed in U:

$$\{x \in U \,\vdots\, u(x) = M\} = u^{-1}(\{M\})$$

shows that $\{u = M\}$ is the preimage of a closed set (under a continuous map), and hence is closed. Thus U connected $\implies U = \{x \in U \,\vdots\, u(x) = M\}$. □

Proof. (ii) $\partial U \subset \bar{U}$, so $m = \max_{\partial U}\{u\} \leqslant \max_{\bar{U}}\{u\} = M$. To see that $m \geqslant M$, note that $\bar{U}$ is compact, u attains its maximum at some point $x_0 \in \bar{U}$. Then we have

case i) $x_0 \in \partial U$. Then $m \geqslant M$ follows immediately.

case ii) $x_0 \in \bar{U} \sim \partial U$. Then $x_0 \in U \implies u$ is constant on U by (i), so $m = M$.

□

Remark. This theorem has an analogous version for minima (proved analogously).

Remark. Suppose U is connected and for some $g \geqslant 0$, $u \in C^2(U) \cap C(\bar{U})$ satisfies

$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

Then $g(x) > 0$ *somewhere* on ∂U implies $u(x) > 0$ *everywhere* on U.

The reasoning behind this is as follows: $g \geqslant 0$, so the lowest $\min_{\bar{U}}\{u\}$ can be is 0, because the minimum is attained on the boundary ∂U, where $u = g$. Then $u(x_0) = 0$ for $x_0 \in U$ implies $u \equiv 0$ by (i). But then the assumption that $g(x) > 0$ for some $x \in \partial U$ gives

$$0 \equiv u = g > 0 \quad \text{on } \partial U \quad ↯.$$

The maximum principle also provides an easy proof for the uniqueness of the solution to Poisson's equation.

Theorem 4. Suppose $g \in C(\partial U)$, $f \in C(U)$. Then there is at most one solution $u \in C^2(U) \cap C(\bar{U})$ to the boundary-value problem

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

Proof. Suppose u, v satisfy $-\Delta u = f, u = g$ and $-\Delta v = f, v = g$. Then $w = u - v$ and $w' = v - u$ are harmonic. Hence $w = w' = g - g \equiv 0$ on ∂U, so

$$\max_{\bar{U}} w = \max_{\bar{U}} w' = \min_{\bar{U}} w = 0 \qquad \Longrightarrow \qquad u \equiv v.$$

□

Theorem 5. If $u \in C(U)$ satisfies $u\,(x) = ⨍_{\partial B(x,r)} u\,dS = ⨍_{B(x,r)} u\,dy$, $\forall B\,(x, r) \subset U$, then $u \in C^\infty(U)$. Consequently, u harmonic $\Longrightarrow$ $u \in C^\infty(U)$.

Proof. Let $\eta \in C^\infty(\mathbb{R}^n)$ be the standard mollifier defined by

$$\eta(x) = \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & |x| < 1 \\ 0 & |x| \geqslant 1 \end{cases},$$

where the constant $C > 0$ is selected such that $\int_{\mathbb{R}^n} \eta\,dx = 1$. Note that $spt(\eta) = B(0, 1)$, and that η is a radial function.

Fix $\varepsilon > 0$ and define

$$\eta_\varepsilon\,(x) = \tfrac{1}{\varepsilon^n} \eta\left(\tfrac{x}{\varepsilon}\right)$$

so that $\eta_e \in C^\infty$, $\int_{\mathbb{R}^n} \eta_\varepsilon dx = 1$, and $spt\,(\eta_\varepsilon) \subset B\,(0, \varepsilon)$.

Now set

$$u^\varepsilon = \eta_\varepsilon * u \quad \text{in } U_\varepsilon = \left\{ x \in U \,\vdots\, dist\,(x, \partial U) > \varepsilon \right\}$$

so that $u^\varepsilon = \int_U \eta_\varepsilon\,(x - y)\,u\,(y)\,dy = \int_{B(0,\varepsilon)} \eta_\varepsilon\,(y)\,u\,(x - y)\,dy$ for $x \in U_\varepsilon$.

We show $u \equiv u^\varepsilon$ on U_ε. For $x \in U_\varepsilon$,

$$\begin{aligned}
u^\varepsilon(x) &= \int_U \eta_\varepsilon(x-y)\,u(y)\,dy \\
&= \int_U \tfrac{1}{\varepsilon^n}\eta\left(\tfrac{x-y}{\varepsilon}\right) u(y)\,dy \\
&= \int_{B(x,\varepsilon)} \tfrac{1}{\varepsilon^n}\eta\left(\tfrac{|x-y|}{\varepsilon}\right) u(y)\,dy \\
&= \int_0^\varepsilon \tfrac{1}{\varepsilon^n}\eta\left(\tfrac{r}{\varepsilon}\right)\left(\int_{\partial B(x,\varepsilon)} u\,dS\right) dr \\
&= u(x)\int_0^\varepsilon \tfrac{1}{\varepsilon^n}\eta\left(\tfrac{r}{\varepsilon}\right) n\alpha(n)\,r^{n-1}dr \\
&= u(x)\int_0^\varepsilon \eta_\varepsilon(r)\,n\alpha(n)\,r^{n-1}dr \\
&= u(x)\int_{B(0,\varepsilon)} \eta_\varepsilon dy \\
&= u(x)
\end{aligned}$$

Thus $u^\varepsilon \equiv u$ in U_ε, and since $u^\varepsilon \in C^\infty$ by construction, this shows $u \in C^\infty(U_\varepsilon), \forall \varepsilon > 0$. Thus $u \in C^\infty(U)$. ☐

Remark. Somehow, the algebraic structure of Laplace's equation $\Delta u = \sum_{i=1}^n u_{x_i x_i} = 0$ leads to the analytic deduction that all the partial derivatives of u exist, even those which do not appear in the PDE.

Theorem 6. (Estimates on derivatives). u is harmonic in $U \implies |D^\alpha u(x_0)| \leqslant \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}$, $\forall B(x_0,r) \subset U$ and for every multiindex α with $|\alpha| = k$. C^k is given here by

$$C_0 = \frac{1}{\alpha(n)},\ C_k = \frac{\left(2^{n+1}nk\right)^k}{\alpha(n)} \quad (k = 1, 2, \ldots)$$

Theorem 7. (Liouville). Let $u : \mathbb{R}^n \to \mathbb{R}$ be harmonic. Then u bounded $\implies$ u constant.

Proof. Fix $x_0 \in \mathbb{R}^n$ and $r > 0$. Now by the previous theorem, u harmonic on $B(x_0,r)$ gives

$$\begin{aligned}
|Du(x_0)| &\leqslant \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x_0,r))} \\
&\leqslant \frac{C_1}{r}\alpha(n)\|u\|_{L^\infty(B(x_0,r))} \xrightarrow{r\to\infty} 0
\end{aligned}$$

Thus $|Du(x_0)| \equiv 0$, and hence u is constant. ☐

Theorem 8. Suppose $f \in C_c^2(\mathbb{R}^n), n \geqslant 3$. Then any bounded solution of $-\Delta u = f$ in $\mathbb{R}^n$ has the form

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy + C \quad (x \in \mathbb{R}^n)$$

for some constant C.

Proof. For $n \geqslant 3$, we have $\Phi(x) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} \xrightarrow{x\to\infty} 0$.

This shows that $u(x) = \int_{\mathbb{R}} \Phi(x-y) f(y)\, dy$ is a *bounded* solution of $-\Delta u = f$ in $\mathbb{R}^n$. Now if v is some other solution, define $w := v - u$. Then

$$\Delta w = \Delta(v-u) = \Delta v - \Delta u = (-f) - (-f) = 0$$

shows w is harmonic. Hence by Liouville's theorem, w is constant. Thus $w = v - u = C$

$$\implies v = u + C = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy + C$$

□

Theorem 9. (Analyticity) u is harmonic in $U \implies u$ analytic in U. That is, u may be represented at x_0 by the Taylor series

$$\sum_\alpha \tfrac{D^\alpha u(x_0)}{\alpha!}(x-x_0)^\alpha, \text{ which converges for } |x-x_0| < \tfrac{\mathrm{dist}(x_0,\partial U)}{2^{n+4}n^3 e}.$$

Theorem 10. (Harnack's Inequality) For each connected open set $V \subset\subset U$, $\exists C > 0$ such that $\sup_V u \leqslant C \inf_V u$, for all nonnegative harmonic functions u in U.

Proof. Let $r = \frac{1}{4} dist(V, \partial U)$. Choose $x, y \in V$ with $|x-y| \leqslant r$. Then

$$\begin{aligned} u(x) &= ⨍_{B(x,2r)} u\, dz \\ &= \tfrac{1}{\alpha(n)(2r)^n} \int_{B(x,2r)} u\, dz \\ &\geqslant \tfrac{1}{2^n} \tfrac{1}{\alpha(n) r^n} \int_{B(y,r)} u\, dz \\ &= \tfrac{1}{2^n} ⨍_{B(y,r)} u\, dz \\ &= \tfrac{1}{2^n} u(y) \end{aligned}$$

$\implies 2^n u(y) \geqslant u(x) \geqslant \frac{1}{2^n} u(y)$ if $x, y \in V$ and $|x-y| \leqslant r$.

Since V is connected and $\bar{V}$ is compact, we can cover $\bar{V}$ by a chain of finitely many balls $\{B_j\}_{j=1}^N$, each of which has radius $\frac{r}{2}$, and such that $B_i \cap B_{i+1} \neq \varnothing$, $\forall i = 1, \ldots N$. Then $u(x) \geqslant \frac{1}{2^{nN}} u(y)$, $\forall x, y \in V$. □

VI. Derivation of Green's Function

In this section we develop a solution of Poisson's equation $-\Delta u = f$ in U, subject to the Dirichlet condition

$$u = g \quad \text{on } \partial U.$$

For convenience, we also make the assumption that $U \subset \mathbb{R}^n$ is open and bounded, and that ∂U is C^1.

Start with an arbitrary function $u \in C^2(\bar{U})$. Fix an $x \in U$, and then fix $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. Define $V_\varepsilon := U \sim B(x, e)$. Consider $u(y)$ and $\Phi(y - x)$ on V_ε. Recall (one of many) Green's formula:

$$\int_U u\Delta v - v\Delta u \, dx = \int_{\partial U} u\tfrac{\partial v}{\partial \nu} - v\tfrac{\partial u}{\partial \nu} dS$$

Applying this formula to $u(y)$ and $\Phi(y - x)$ on V_ε gives

$$\int_{V_\varepsilon} u(y)\, \Delta\Phi(y - x)\, dy - \int_{V_\varepsilon} \Phi(y - x)\, \Delta u(y)\, dy \tag{\dagger}$$
$$= \int_{\partial V_\varepsilon} u(y)\, \tfrac{\partial \Phi}{\partial \nu}(y - x) - \Phi(y - x)\, \tfrac{\partial u}{\partial \nu}(y)\, dS(y)$$

where ν denotes the outward unit normal vector on ∂V_ε.

Next observe that $\partial V_\varepsilon = \partial U \sqcup \partial B$ (disjoint union), so we may separate the integral over ∂V_ε into two pieces accordingly. Now on ∂B we have the estimates

$$\left| \int_{\partial B} \Phi(y - x)\, \tfrac{\partial u}{\partial \nu}(y)\, dS(y) \right| \leqslant C\varepsilon^{n-1} \max_{\partial B} |\Phi| \xrightarrow{\varepsilon \to 0} 0$$

and

$$\left| \int_{\partial B} u(y)\, \tfrac{\partial \Phi}{\partial \nu}(y - x)\, dS(y) \right| = \fint_{\partial B} u(y)\, dS(y) \xrightarrow{\varepsilon \to 0} u(x).$$

Therefore, when we let $\varepsilon \to 0$, the right-hand side of (†)) becomes

$$\int_{\partial U} u(y)\, \tfrac{\partial \Phi}{\partial \nu}(y - x) - \Phi(y - x)\, \tfrac{\partial u}{\partial \nu}(y)\, dS(y) + u(x).$$

Moreover, $\Delta\Phi(y - x) = 0$ for $y \neq x$, so $V_\varepsilon = U \sim B(x, e)$ implies

$$\int_{V_\varepsilon} u(y)\, \Delta\Phi(y - x)\, dS(y) = \int_{V_\varepsilon} 0\, dS = 0, \ \forall \varepsilon > 0.$$

Thus letting $\varepsilon \to 0$ transforms (†) from

$$\int_{V_\varepsilon} u(y)\,\Delta\Phi(y-x)\,dy - \int_{V_\varepsilon} \Phi(y-x)\,\Delta u(y)\,dy = \int_{\partial V_\varepsilon} u(y)\,\tfrac{\partial\Phi}{\partial\nu}(y-x) - \Phi(y-x)\,\tfrac{\partial u}{\partial\nu}(y)\,dS(y)$$

into the simpler expression

$$-\int_U \Phi(y-x)\,\Delta u(y)\,dy = \int_{\partial U} u(y)\,\tfrac{\partial\Phi}{\partial\nu}(y-x) - \Phi(y-x)\,\tfrac{\partial u}{\partial\nu}(y)\,dS(y) + u(x).$$

Rearranging and substituting, we obtain the following expression for u:

$$\begin{aligned} u(x) &= \int_{\partial U} \left[\Phi(y-x)\,\tfrac{\partial u}{\partial\nu}(y) - u(y)\,\tfrac{\partial\Phi}{\partial\nu}(y-x)\right] dS(y) - \int_U \Phi(y-x)\,\Delta u(y)\,dy \\ &= \int_{\partial U} \left[\Phi(y-x)\,\tfrac{\partial u}{\partial\nu}(y) - g(y)\,\tfrac{\partial\Phi}{\partial\nu}(y-x)\right] dS(y) + \int_U \Phi(y-x)\,f(y)dy. \end{aligned}$$

This identity is valid $\forall x \in U$, $\forall u \in C^2(\bar{U})$, so we could solve for u if we knew $\frac{\partial u}{\partial\nu}$ on ∂U. Unfortunately, we *don't* know what the normal derivative $\frac{\partial u}{\partial\nu}$ is along ∂U. Therefore, we introduce (for fixed x) a "*corrector* function" $\phi^x = \phi^x(y)$ that solves the boundary value problem

$$\begin{cases} \Delta\phi^x = 0 & \text{in } U \\ \phi^x = \Phi(y-x) & \text{on } \partial U \end{cases}$$

We apply Green's formula $\int_U u\Delta v - v\Delta u\,dx = \int_{\partial U} u\frac{\partial v}{\partial\nu} - v\frac{\partial u}{\partial\nu}dS$ to ϕ^x and obtain

$$\int_U \left[u(y)\,\Delta\phi^x(y) - \phi^x(y)\,\Delta u(y)\right] dy = \int_{\partial U} \left[u(y)\,\tfrac{\partial\phi^x}{\partial\nu}(y) - \phi^x(y)\,\tfrac{\partial u}{\partial\nu}(y)\right] dS(y).$$

Now by definition of ϕ^x, $\Delta\phi^x = 0$ in U and $\phi^x = \Phi(y-x)$ on ∂U, so this expression is really

$$\begin{aligned} &-\int_U \phi^x(y)\,\Delta u(y)\,dy = \int_{\partial U} \left[u(y)\,\tfrac{\partial\phi^x}{\partial\nu}(y) - \Phi(y-x)\,\tfrac{\partial u}{\partial\nu}(y)\right] dS(y) \\ \implies &\int_{\partial U} \Phi(y-x)\,\tfrac{\partial u}{\partial\nu}(y)\,dS(y) = \int_U \phi^x(y)\,\Delta u(y)\,dy + \int_{\partial U} u(y)\,\tfrac{\partial\phi^x}{\partial\nu}(y)\,dS(y). \end{aligned}$$

With the $\frac{\partial u}{\partial\nu}$ term thus isolated, we substitute back into the expression for $u(x)$:

$$\begin{aligned} u(x) &= \int_U \phi^x(y)\,\Delta u(y)\,dy + \int_{\partial U} u(y)\,\tfrac{\partial\phi^x}{\partial\nu}(y)\,dS(y) \\ &\qquad - \int_{\partial U} u(y)\,\tfrac{\partial\Phi}{\partial\nu}(y-x)\,dS(y) - \int_U \Phi(y-x)\,\Delta u(y)\,dy \\ &= \int_U \left[\phi^x(y) - \Phi(y-x)\right] \Delta u(y)\,dy + \int_{\partial U} u(y)\left[\tfrac{\partial\phi^x}{\partial\nu}(y) - \tfrac{\partial\Phi}{\partial\nu}(y-x)\right] dS(y) \end{aligned}$$

Definition 6. The *Green's function* for the region U is

$$G(x,y) := \Phi(y-x) - \phi^x(y), \qquad x, y \in U,\ x \neq y.$$

Note that G is harmonic for $x \neq y$.

Continuing from above, this terminology yields

$$u(x) = -\int_U G(x,y)\,\Delta u(y)\,dy - \int_{\partial U} u(y)\,\tfrac{\partial G}{\partial \nu}(x,y)\,dS(y),$$

where $\frac{\partial G}{\partial \nu}(x,y) := D_y G(x,y)\cdot \bar{\nu}(y)$ is the outer normal derivative of G with respect to y.

We summarize these results to obtain

Theorem 11. Suppose that $u \in C^2(\bar{U})$ solves the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

for given f, g. Then

$$u(x) = \int_U G(x,y)\,f(y)\,dy - \int_{\partial U} g(y)\,\tfrac{\partial G}{\partial \nu}(x,y)\,dS(y) \qquad (x \in U). \tag{*}$$

Lemma 2. (Symmetry of Green's function). $\forall x, y \in U, x \neq y$, we have $G(y,x) = G(x,y)$.

Proof. Fix $x, y \in U, x \neq y$. For $z \in U$, let us denote

$$v(z) := G(x,z) \qquad \text{and} \qquad w(z) := G(y,z).$$

Then (i) $\Delta v(z) = 0$ for $z \neq y$, (ii) $\Delta w(z) = 0$ for $z \neq y$, and (iii) $v = w = 0$ on ∂U. Choose $\varepsilon > 0$ so small that $\partial B(x,\varepsilon) \cap \partial B(y,\varepsilon) = \varnothing$. Now apply Green's Identity on $V := U \sim (B(x,\varepsilon) \cup B(y,\varepsilon))$ to get

$$\int_{\partial B(x,\varepsilon)} \left[\tfrac{\partial v}{\partial \nu} w - \tfrac{\partial w}{\partial \nu} v\right] dS(z) = \int_{\partial B(y,\varepsilon)} \left[\tfrac{\partial w}{\partial \nu} v - \tfrac{\partial v}{\partial \nu} w\right] dS(z)$$

where ν is the inward-pointing unit vector field on $\partial B(x,\varepsilon) \cap \partial B(y,\varepsilon)$.

Now w is smooth near x, so

$$\int_{\partial B(x,\varepsilon)} \tfrac{\partial w}{\partial \nu} v\,dS(z) \leqslant C\varepsilon^{n-1} \sup_{\partial B(x,\varepsilon)} |v| \xrightarrow{\varepsilon \to 0} 0.$$

On the other hand, $v(z) = \Phi(y-x) - \phi^x(z)$ where ϕ^x is smooth in U, so

$$\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \tfrac{\partial v}{\partial \nu} w\,dS(z) = \lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \tfrac{\partial \Phi}{\partial \nu}(x-y)\,w(z)\;dS(z) = w(x),$$

by calculations as in the derivation of the solution for Poisson's equation. Now

$$\begin{array}{ccccccc} \displaystyle\int_{\partial B(x,\varepsilon)} \tfrac{\partial v}{\partial \nu} w\,dS(z) & - & \displaystyle\int_{\partial B(x,\varepsilon)} \tfrac{\partial w}{\partial \nu} v\,dS(z) & = & \displaystyle\int_{\partial B(y,\varepsilon)} \tfrac{\partial w}{\partial \nu} v\,dS(z) & - & \displaystyle\int_{\partial B(y,\varepsilon)} \tfrac{\partial v}{\partial \nu} w\,dS(z) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ w(x) & - & 0 & = & v(y) & - & 0 \end{array}$$

shows $G(y,x) = w(x) = v(y) = G(x,y)$. ∎

VII. Green's function for a half-space

Definition 7. We will use the following notation for the half-space in $\mathbb{R}^n$:

$$\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,\vdots\, x_n > 0\}.$$

It is the goal of this section to produce an explicit expression for Green's function for $U = \mathbb{R}^n_+$. Note: Some of the previous results do not directly apply here, as this region is not bounded. Nevertheless, we will be able to find some solutions using these methods — with the caveat that we must check their validity afterwards.

Definition 8. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$, then its *reflection* in the plane $\partial\mathbb{R}^n_+$ is the point

$$\tilde{x} := (x_1, \ldots, -x_n) \notin \mathbb{R}^n_+.$$

Intuitively, the idea is to use the corrector function ϕ^x to "reflect the singularity" out of the domain $\mathbb{R}^n_+$. To that end, we define

$$\phi^x(y) := \Phi(y - \tilde{x}) = \Phi(y_1 - x_1, y_2 - x_2, \ldots, y_{n-1} - x_{n-1}, y_n + x_n) \qquad (x, y \in \mathbb{R}^n_+).$$

Then for $y \in \partial\mathbb{R}^n_+$, y is equidistant from x and $\tilde{x}$. Therefore,

$$|y - x| = |y - \tilde{x}| \implies \Phi(y - x) = \Phi(y - \tilde{x}) \implies \Phi(y - x) = \phi^x(y) \qquad (y \in \partial\mathbb{R}^n_+)$$

because Φ is radial. And since $\tilde{x} \notin \mathbb{R}^n_+$,

$$\Delta\phi^x(y) = \Delta\Phi(y - \tilde{x}) = 0 \qquad (y \in \mathbb{R}^n_+).$$

Thus we have satisfied the requirements for ϕ^x and are justified in making the following definition:

Definition 9. *Green's function for* $\mathbb{R}^n_+$ is

$$G(x, y) := \Phi(y - x) - \Phi(y - \tilde{x}) \qquad (x, y \in \mathbb{R}^n_+, x \neq y)$$

At this point, we pause to compute

$$\begin{aligned}
\tfrac{\partial \Phi}{\partial y_n}(y - x) &= \tfrac{1}{n(n-2)\alpha(n)} \tfrac{\partial}{\partial y_n} \left(\tfrac{1}{|y-x|^{n-2}} \right) \\
&= \tfrac{1}{n(n-2)\alpha(n)} \tfrac{\partial}{\partial y_n} \left[(y_1 - x_1)^2 + (y_2 - x_2)^2 + \ldots + (y_n - x_n)^2 \right]^{\frac{2-n}{2}} \\
&= \tfrac{-1}{n\alpha(n)} \tfrac{1}{2} \left[(y_1 - x_1)^2 + (y_2 - x_2)^2 + \ldots + (y_n - x_n)^2 \right]^{\frac{-n}{2}} \cdot 2\,(y_n - x_n) \\
&= \tfrac{-1}{n\alpha(n)} \tfrac{y_n - x_n}{|y-x|^n}
\end{aligned}$$

so that $\frac{\partial G}{\partial y_n}(x, y) = \frac{\partial \Phi}{\partial y_n}(y - x) - \frac{\partial \Phi}{\partial y_n}(y - \tilde{x}) = \frac{-1}{n\alpha(n)} \left[\frac{y_n - x_n}{|y-x|^n} - \frac{y_n + x_n}{|y-\tilde{x}|^n} \right]$, and hence

$$\tfrac{\partial G}{\partial \nu}(x, y) = -\tfrac{\partial G}{\partial y_n}(x, y) = \tfrac{-2x_n}{n\alpha(n)} \tfrac{1}{|y-x|^n}.$$

If u solves the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U, \end{cases}$$

we can plug in to the previously derived formula (*) to obtain

$$u(x) = \tfrac{-2x_n}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+} \tfrac{g(y)}{|y-x|^n} dy \quad \left(x \in \mathbb{R}^n_+\right).$$

This result is well-known:

Definition 10.
Poisson's formula for $\mathbb{R}^n_+$: $\quad u(x) = \tfrac{-2x_n}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+} \tfrac{g(y)}{|y-x|^n} dy \quad \left(x \in \mathbb{R}^n_+\right).$
Poisson's kernel for $\mathbb{R}^n_+$: $\quad K(x,y) := \tfrac{2x_n}{n\alpha(n)} \tfrac{1}{|x-y|^n}.$

Now as promised, we must validate these results.

Theorem 12. (Poisson's formula for $\mathbb{R}^n_+$). For $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ and

$$u(x) = \tfrac{2x_n}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+} \tfrac{g(y)}{|y-x|^n} dy,$$

u solves the Dirichlet problem on $\mathbb{R}^n_+$:

(i) $u \in C^\infty(\mathbb{R}^n_+) \cup L^\infty(\mathbb{R}^n_+)$,

(ii) $\Delta u = 0$ in $\mathbb{R}^n_+$, and

(iii) $u(x) \xrightarrow{x \to x^0,\, x \in \mathbb{R}^n_+} g(x^0)$, for each $x^0 \in \partial \mathbb{R}^n_+$.

Proof. (i)&(ii) For fixed x, the mapping $y \mapsto G(x,y)$ is harmonic for $y \neq x$. Then by Symmetry Lemma, $x \mapsto G(x,y)$ is harmonic for $y \neq x$. Thus

$$x \mapsto -\tfrac{\partial G}{\partial y_n}(x,y) = \tfrac{2x_n}{n\alpha(n)} \tfrac{1}{|x-y|^n}$$

is a harmonic mapping for $x \in \mathbb{R}^n_+, y \in \partial \mathbb{R}^n_+$.

We use the fact that

$$\int_{\partial \mathbb{R}^n_+} K(x,y) dy = 1$$

for each $x \in \mathbb{R}^n_+$ (calculation omitted).

Thus $g \in L^\infty \implies u$ bounded, as follows:

$$\begin{aligned} |u(x)| &\leqslant \tfrac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \tfrac{|g(y)|}{|y-x|^n} dy \\ &\leqslant \tfrac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \tfrac{\|g\|_\infty}{|y-x|^n} dy \\ &\leqslant \|g\|_\infty \cdot \tfrac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \tfrac{1}{|y-x|^n} dy \\ &= \|g\|_\infty \end{aligned}$$

Since $x \mapsto K(x,y)$ is smooth for $x \neq y$,

$$\Delta u(x) = \int_{\partial\mathbb{R}^n_+} \Delta_x K(x,y)\, g(y)\, dy = 0$$

shows that $u \in C^\infty$. □

Proof. (iii) Fix $x^0 \in \partial\mathbb{R}^n_+$ and $\varepsilon > 0$. Choose $\delta > 0$ small enough that

$$\left|y - x^0\right| < \delta \quad \implies \quad \left|g(y) - g\left(x^0\right)\right| < \varepsilon, \quad \forall y \in \partial\mathbb{R}^n_+.$$

Then if $|x - x^0| < \frac{\delta}{2}$ for $x \in \mathbb{R}^n_+$,

$$\begin{aligned} \left|u(x) - g\left(x^0\right)\right| &= \left|\int_{\partial\mathbb{R}^n_+} K(x,y)\, g(y)\, dy - g\left(x^0\right)\right| \\ &= \left|\int_{\partial\mathbb{R}^n_+} K(x,y)\, g(y)\, dy - \int_{\partial\mathbb{R}^n_+} K(x,y)\, g\left(x^0\right) dy\right| \\ &= \int_{\partial\mathbb{R}^n_+} K(x,y) \left|g(y) - g\left(x^0\right)\right| dy \\ &\leqslant \underbrace{\int_{\partial\mathbb{R}^n_+ \cap B(x^0,\delta)} K(x,y) \left|g(y) - g\left(x^0\right)\right| dy}_{I} \\ &\qquad + \underbrace{\int_{\partial\mathbb{R}^n_+ \sim B(x^0,\delta)} K(x,y) \left|g(y) - g\left(x^0\right)\right| dy}_{J}. \end{aligned}$$

Now for I, $|y - x^0| < \delta$ for $y \in \left(\partial\mathbb{R}^n_+ \cap B\left(x^0, \delta\right)\right)$ gives

$$\begin{aligned} I &= \int_{\partial\mathbb{R}^n_+ \cap B(x^0,\delta)} K(x,y)\left|g(y) - g\left(x^0\right)\right| dy \\ &\leqslant \varepsilon \int_{\partial\mathbb{R}^n_+} K(x,y)\, dy \\ &= \varepsilon. \end{aligned}$$

For J, note that

$$\begin{aligned} &\left|x - x^0\right| < \tfrac{\delta}{2}, \left|y - x^0\right| \geqslant \delta \\ &\qquad \implies \left|y - x^0\right| \leqslant |y - x| + \tfrac{\delta}{2} \leqslant |y - x| + \tfrac{1}{2}\left|y - x^0\right| \\ &\qquad \implies |y - x| \geqslant \tfrac{1}{2}\left|y - x^0\right|. \end{aligned}$$

Thus

$$\begin{aligned} J &\leqslant 2\|g\|_{L^\infty} \int_{\partial\mathbb{R}^n_+ - B(x^0,\delta)} K(x,y)\, dy \\ &\leqslant \frac{2^{n+2} x_n \|g\|_{L^\infty}}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ - B(x^0,\delta)} \left|y - x^0\right|^{-n} dy \quad \xrightarrow{x_n \to 0^+} 0, \end{aligned}$$

and therefore

$$\left|x - x^0\right| < \tfrac{\delta}{2} \quad \implies \quad \left|u(x) - g\left(x^0\right)\right| < \varepsilon.$$

□

References

[Beto] Betounes, D. (1999) *Partial Differential Equations for Computational Science.* Springer-Verlag.

[BoyDi] Boyce, W.E., and DiPrima, R.C. (1997) *Elementary Differential Equations and Boundary Value Problems.* John Wiley.

[Evans] Evans, L.C. (1998) *Partial Differential Equations.* AMS (GSM v.19).

[GilTr] Gilbarg, D., and Trudinger, N.S. (1983) *Elliptic Partial Differential Equations of Second Order (2nd ed).* Springer-Verlag.