1. Consider the pendulum equation $x'' = -\sin x$.

(a) Prove it is a Hamiltonian system.

First make the standard transformation into a system of first-order equations: let $y = x'$ so that $y' = x''$ and the system becomes

$$\begin{align*}
x' &= y \\
y' &= -\sin x
\end{align*}$$

Then take $H(x, y) = \frac{y^2}{2} - \cos x$. Then

$$\frac{\partial H}{\partial y} = y \quad \text{and} \quad -\frac{\partial H}{\partial x} = -\sin x,$$

so we have

$$\begin{align*}
x' &= \frac{\partial H}{\partial y} \\
y' &= -\frac{\partial H}{\partial x}
\end{align*}$$

(b) Find the general solution (in integral form) and sketch the phase plane.

We proceed by integrating each side of $x'' = -\sin x$ to obtain

$$x'(t) = x_0 - \int_{t_0}^{t} \sin x(s) \, ds.$$ 

Then, considering $F(t) := x_0 - \int_{t_0}^{t} \sin x(s) \, ds$, we integrate again and obtain

$$x(t) = x_1 + \int_{t_0}^{t} F(u) \, du$$

$$= x_1 + \int_{t_0}^{t} \left( x_0 - \int_{u_0}^{u} \sin x(s) \, ds \right) \, du$$

$$= x_1 + \int_{t_0}^{t} x_0 \, du - \int_{t_0}^{t} \int_{u_0}^{u} \sin x(s) \, ds \, du$$

$$= x_1 + x_0 (t - t_0) - \int_{t_0}^{t} \int_{u_0}^{u} \sin x(s) \, ds \, du$$

This is verified to be correct by differentiating twice with respect to $t$ and applying the Fundamental Theorem of Calculus each time.
Figure 1. $x'' = \sin x$.

$x' = y$
$y' = \sin(x)$
2. For \( f \in C^1(\mathbb{R}^n) \) and each \( x_0 \in \mathbb{R}^n \), prove the initial value problem
\[
\begin{align*}
x' &= \frac{f(x)}{1 + f(x)}, \\
x(0) &= x_0
\end{align*}
\]
has a unique solution for all \( t \in \mathbb{R} \).

It would be nice to use Gronwall’s inequality, but this is not possible as we have neither the continuity of \( 1/(1 + f(x)) \) nor a relation between \( x(t) \) and \( f(x) \) or \( 1/(1 + f(x)) \), so we will have to use the autonomy of the system instead. Define
\[
F(t, x) = F(x) := \frac{f(x)}{1 + f(x)} \quad \text{and} \quad E := \{(t, x) : f(x(t)) \neq -1\}
\]
so that \( F \) has continuous partial derivatives
\[
\frac{\partial F}{\partial x_i} = \frac{\frac{\partial f(x)}{\partial x_i}}{(1 + f(x))^2}
\]
throughout \( E \). If \( E = \emptyset \) then the system is degenerate, so let \( x_0 \in E \). Then if the complement of \( E \) is dense in \( \mathbb{R}^n \), \( E = \emptyset \) by continuity, so we can find some open connected domain \( B \) such that \( x_0 \in B \subseteq E \).

Note that \( F(t, x) \) is defined and continuous in \( B \). By the Existence and Uniqueness Theorem, there is a solution \( x = \varphi(t) \) satisfying the system
\[
\begin{align*}
x' &= F(t, x) \\
x(0) &= x_0
\end{align*}
\]
and defined in some neighbourhood of \( (0, x_0) \). Now note that \( x' = F(t, x) = F(x) \) is an autonomous equation! This fact may be exploited. Let \( s \in \mathbb{R} \). Then
\[
\frac{\partial \varphi}{\partial t}(t + s) \bigg|_{t=t_0} = \frac{\partial \varphi}{\partial t}(t) \bigg|_{t=t_0+s} = F(\varphi(t)) \bigg|_{t=t_0+s} = F(\varphi(t + s)) \bigg|_{t=t_0}.
\]
We have just shown that if \( \varphi(t) \) is a solution to the autonomous system, then \( \varphi(t + s) \) will also be a solution, for any \( s \in \mathbb{R} \). This makes it clear that the solution \( \varphi(t) \) must exist for all time. Otherwise, if it had some bounded maximal interval of existence \( (t_1, t_2) \), we would have a contradiction: for \( t_0 \in (t_1, t_2) \),
\[
\frac{\partial \varphi}{\partial t}(t_0) = F(\varphi(t_0)),
\]
by definition of interval of existence. But then letting \( s = (t_2 - t_0) \), the previous argument shows that
\[
\frac{\partial \varphi}{\partial t}(t_2) = \frac{\partial \varphi}{\partial t}(t_0 + s) = F(\varphi(t_0 + s)) = F(\varphi(t_2)),
\]
contradicting the maximality of the interval. Indeed, taking \( s = \alpha(t_2 - t_0) \) for any \( \alpha > 0 \) shows that \( \varphi(t) \) is a valid solution for all positive time. A symmetric argument shows \( \varphi(t) \) is also a valid solution for all negative time.
3. Use Liapunov’s method to determine the stability of the critical point \((0,0,0)\) of the system

\[
\begin{align*}
    x'_1 &= -2x_2 + x_2x_3 - x_1^3, \\
    x'_2 &= x_1 - x_1x_3 - x_2^3, \\
    x'_3 &= x_1x_2 - x_3^3.
\end{align*}
\]

We try the Liapunov candidate

\[ V(x) = ax_1^2 + bx_2^2 + cx_3^2. \]

Then

\[
\begin{align*}
  \dot{V} &= 2ax_1\dot{x}_1 + 2bx_2\dot{x}_2 + 2cx_3\dot{x}_3 \\
  &= -4ax_1x_2 + 2ax_1x_2x_3 - 2ax_1^4 + 2bx_1x_2 - 2bx_1x_2x_3 - 2bx_3^4 + 2cx_1x_2x_3 - 2cx_3^4 \\
  &= -2(ax_1^4 + bx_2^4 + cx_3^4) + 2(a - b + c)x_1x_2x_3 - 2(2a - b)x_1x_2
\end{align*}
\]

To make things simpler, let \(b = 2a\) to make the last term vanish. Then make \(c = a\) to make the second term vanish. For definiteness, use \(a = c = 1, b = 2\). Thus, the Liapunov function

\[ V(x, y) = x_1^2 + 2x_2^2 + x_3^2 \]

is positive definite and

\[ V(x, y) = -2(x_1^4 + x_2^4 + x_3^4) < 0, \quad \text{and} \quad \dot{V}(0,0) = 0, \]

shows that \(V\) is negative definite. This is enough to show that \((0,0,0)\) is stable, but we can go even further and define

\[ \psi(||x||) = \sqrt{2}||x||. \]

Then for \(||x|| < 1\) we have

\[
\begin{align*}
  \psi(||x||) &= \sqrt{2}||x|| \\
  &= \sqrt{2}\sqrt{x_1^2 + x_2^2 + x_3^2} \\
  &= \sqrt{2x_1^2 + 2x_2^2 + 2x_3^2} \\
  &\geq 2x_1^2 + 2x_2^2 + 2x_3^2 \\
  &\geq x_1^2 + x_2^2 + x_3^2 \\
  &= V(x),
\end{align*}
\]

so that \(V\) is descreasing. Therefore, \((0,0,0)\) is asymptotically stable, by Thm 5.4.2.
Figure 2. A phase portrait for the system in Problem 3.

All flow lines point toward the origin.
4. Consider the linear system

\[ x'' + p(t)x' + q(t)x = 0, \]

where \( p \) and \( q \) are real-valued and continuous. Let \( y_1, y_2 \) be a real-valued fundamental pair of solutions. Show that \( y_2 \) must vanish between any two consecutive zeroes of \( y_1 \).

Let \( t_1, t_2 \) be consecutive zeroes of \( y_1 \). Then

\[ W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t) \]

is continuous, as a sum of products of continuous functions. Then

\[ W(t_1) = -y_1'(t_1)y_2(t_1) \quad \text{and} \quad W(t_2) = -y_1'(t_2)y_2(t_2). \]

So none of \( y_1'(t_1), y_2(t_1), y_1'(t_2), y_2(t_2) \) can be 0, or else \( W \) would be 0, and then \( y_1, y_2 \) couldn’t be linearly independent, by Thm 2.3.2.

Suppose that \( W(t_1) > 0 \) and that \( y_2(t_1) > 0 \). Then \( y_1'(t_1) < 0 \), that is, \( y_1(t) \) is decreasing as it passes through \( t_1 \). Then, since \( y_1(t) \) is continuous and has no zeroes between \( t_1 \) and \( t_2 \), \( y_1(t) \) must be increasing as it passes through \( t_2 \) by basic calculus. I.e., \( y_1'(t_2) > 0 \).

Note that we cannot have \( W(t_2) < 0 \): since \( W(t) \) is continuous, the Intermediate Value Theorem would imply the existence of some \( t_0 \in (t_1, t_2) \) for which \( W(t_0) = 0 \), which would be a contradiction as described above. Thus we have

\[ W(t_2) = -y_1'(t_2)y_2(t_2) > 0. \]

Since \( y_1'(t_2) > 0 \), this implies \( y_2(t_1) < 0 \). So \( y_2(t) \) has changed sign somewhere between \( t_1 \) and \( t_2 \); by the IVT again, there must be a \( t_0 \in (t_1, t_2) \) for which \( y_2(t_0) = 0 \).

This argument has given the desired result for the case when \( W(t_1) > 0 \) and \( y_2(t_1) > 0 \), but it is clear that a similar argument works just as well if we take \( y_2(t_1) < 0 \), and for the two cases when \( W(t_1) < 0 \). So in any case, we can find a zero of \( y_2 \) between any two consecutive zeroes of \( y_1 \).