ON γ-VECTORS SATISFYING KRUSKAL-KATONA INEQUALITIES

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Abstract. We present examples of flag homology spheres whose γ-vectors satisfy the Kruskal-Katona inequalities. This includes several families of well-studied simplicial complexes, including Coxeter complexes and the simplicial complexes dual to the associahedron and to the cyclohedron. In another direction, we show that if a flag $(d-1)$-sphere has at most $2d+2$ vertices its γ-vector satisfies the Kruskal-Katona inequalities. We conjecture that if Δ is a flag homology sphere then $\gamma(\Delta)$ satisfies the Kruskal-Katona inequalities. This conjecture is a significant refinement of Gal's conjecture, which asserts that such γ-vectors are nonnegative.

1. Introduction

In [3] Gal gave counterexamples to the real-root conjecture for flag spheres and conjectured a weaker statement which still implies the Charney-Davis conjecture. The conjecture is phrased in terms of the so-called *γ-vector*.

Conjecture 1.1 (Gal)**.** [3, Conjecture 2.1.7] *If Δ is a flag homology sphere then $\gamma(\Delta)$ is nonnegative.*

This conjecture is known to hold for the order complex of a Gorenstein* poset [5], all Coxeter complexes (see [13], and references therein), and for the (dual simplicial complexes of the) "chordal nestohedra" of [9]—a class containing the associahedron, permutahedron, and other well-studied polytopes.

If Δ has a nonnegative γ-vector, one may ask what these nonnegative integers count. In certain cases (the type A Coxeter complex, say), the γ-vector has a very explicit combinatorial description. We will exploit such descriptions to show that not only are these numbers nonnegative, but they satisfy certain non-trivial inequalities known as the *Kruskal-Katona inequalities.* Put another way, such a γ-vector is the f-vector of a simplicial complex. Our main result is the following.

Theorem 1.2. *The γ-vector of Δ satisfies Kruskal-Katona inequalities for each of the following classes of flag spheres:*

(a) *Δ is a Coxeter complex.*
(b) *Δ is the simplicial complex dual to an associahedron.*

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(c) Δ *is the simplicial complex dual to a cyclohedron (type B associahedron).*

Note that the type A Coxeter complex is the dual simplicial complex to the permutahedron and the type B Coxeter complex is dual to the type B permutahedron.

We prove Theorem 1.2 by imposing a particular poset structure on the combinatorial objects enumerated by $\gamma(\Delta)$. The Kruskal-Katona inequalities then follow from a result of Wegner [14].

In a different direction, we are also able to show that if Δ is a flag sphere with few vertices relative to its dimension, then its γ-vector satisfies the Kruskal-Katona inequalities.

Theorem 1.3. *Let Δ be a $(d-1)$-dimensional flag sphere with at most $2d+2$ vertices, i.e., with $\gamma_1(\Delta) \leq 2$. Then $\gamma(\Delta)$ satisfies the Kruskal-Katona inequalities. Moreover, all possible γ-polynomials, i.e., $1, 1+t, 1+2t$, and $1+2t+t^2$ occur as $\gamma(\Delta; t)$ for some flag sphere Δ.*

Theorem 1.3 is proved by characterizing the structure of such flag spheres.

Computer evidence suggests that Theorems 1.2 and 1.3 may be enlarged significantly. We make the following strengthening of Gal's conjecture.

Conjecture 1.4. *If Δ is a flag homology sphere then $\gamma(\Delta)$ satisfies the Kruskal-Katona inequalities.*

This conjecture is true for flag homology 3- (or 4-) spheres. Indeed, Gal showed that $\gamma_2(\Delta) \leq \gamma_1(\Delta)^2/4$ must hold for flag homology 3- (or 4-) spheres [3], which implies the Kruskal-Katona inequality $\gamma_2(\Delta) \leq \binom{\gamma_1(\Delta)}{2}$ for the case $\gamma_1(\Delta) \geq 2$. The case $\gamma_1(\Delta) \leq 1$ follows from Theorem 1.3.

In Section 2 we review some definitions and known results. Section 3 is devoted to the proof of Theorem 1.2, while Section 4 is given to Theorem 1.3.

2. Terminology

A *simplicial complex* Δ on a vertex set V is a collection of subsets F of V, called *faces*, which contains all the singletons and such that if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$. The dimension of a face F is $\dim F = |F| - 1$, in particular $\dim \emptyset = -1$. The dimension of Δ, denoted by $\dim \Delta$, is the maximum of the dimensions of its faces. For $A \subseteq V$, $\Delta[A]$ is the *induced subcomplex* of Δ on A, consisting of all faces F of Δ such that $F \subseteq A$. The *antistar* $\mathrm{ast}(v)$ of a vertex $v \in V$ is the induced subcomplex $\Delta[V - \{v\}]$. We say that Δ is *flag* if all the minimal subsets of V which are not in Δ have size 2; equivalently $F \in \Delta$ if and only if all the two element subsets of F are in Δ.

We say that Δ is a sphere if its geometric realization is homeomorphic to a sphere. The *link* $\mathrm{lk}(F) = \mathrm{lk}_\Delta(F)$ of a face F of Δ is the set of all $G \in \Delta$ such that $F \cup G \in \Delta$ and $F \cap G = \emptyset$. We say that Δ is a *homology sphere* if for every face $F \in \Delta$, $\mathrm{lk}(F)$ is homologous to the $(\dim \Delta - |F|)$-dimensional sphere. In particular, if Δ is a sphere then Δ is a homology sphere.

The *f-polynomial* of a $(d-1)$-dimensional simplicial complex Δ is the generating function for the dimensions of the faces of the complex:

$$f(\Delta; t) := \sum_{F \in \Delta} t^{\dim F + 1} = \sum_{0 \leq i \leq d} f_i(\Delta) t^i.$$

The *f-vector*

$$f(\Delta) := (f_0, f_1, \ldots, f_d)$$

is the sequence of coefficients of the f-polynomial.

The *h-polynomial* of Δ is a transformation of the f-polynomial:

$$h(\Delta; t) := (1-t)^d f(\Delta; t/(1-t)) = \sum_{0 \leq i \leq d} h_i(\Delta) t^i,$$

and the *h-vector* is the corresponding sequence of coefficients,

$$h(\Delta) := (h_0, h_1, \ldots, h_d).$$

Though they contain the same information, often the h-polynomial is easier to work with than the f-polynomial. For instance, if Δ is a homology sphere, then the *Dehn-Sommerville relations* guarantee that the h-vector is symmetric, i.e., $h_i = h_{d-i}$ for all $0 \leq i \leq d$.

When referring to the f- or h-polynomial of a simple polytope P, we mean the f- or h-polynomial of the boundary complex of its dual, denoted by Δ_P. So, for instance, we refer to the h-vector of the type A Coxeter complex and the permutahedron interchangably.

Whenever a polynomial of degree d has symmetric integer coefficients, it has an expansion in the basis $\{t^i(1+t)^{d-2i} : 0 \leq i \leq d/2\}$. Specifically, if Δ is a $(d-1)$-dimensional homology sphere then there exist integers $\gamma_i(\Delta)$ such that

$$h(\Delta; t) = \sum_{0 \leq i \leq d/2} \gamma_i(\Delta) t^i (1+t)^{d-2i}.$$

We refer to the sequence $\gamma(\Delta) := (\gamma_0, \gamma_1, \ldots)$ as the *γ-vector* of Δ, and the corresponding generating function $\gamma(\Delta; t) = \sum \gamma_i t^i$ is the *γ-polynomial.* Our goal is to show that under the hypotheses of Theorems 1.2 and 1.3 the γ-vector for Δ is seen to be the f-vector for some other simplicial complex.

A result of Schützenberger, Kruskal and Katona (all independently), characterizes the f-vectors of simplicial complexes as follows. (See [12, Ch. II.2].) By convention we call the conditions characterizing these f-vectors the *Kruskal-Katona inequalities*.

Given a pair of integers a and i there is a unique expansion:

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j},$$

where $a_i > a_{i-1} > \cdots > a_j \geq j$. With this in mind, define

$$a^{(i)} = \binom{a_i+1}{i} + \binom{a_{i-1}+1}{i-1} + \cdots + \binom{a_j+1}{j}, \quad 0^{(i)} = 0.$$

Theorem 2.1 (Katona, Kruskal, Schützenberger)**.** *An integer vector* $(f_0, f_1 \ldots)$ *is the* f*-vector of a simplicial complex if and only if:*

(a) $f_0 = 1$,
(b) $f_i \geq 0$,
(c) $f_{i+1} \leq f_i^{(i)}$ *for* $i = 1, 2 \ldots$.

We will use the Kruskal-Katona inequalities directly for Theorem 1.3 and for checking the Coxeter complexes of exceptional type in part (a) of Theorem 1.2. For the remainder of Theorem 1.2 we appeal to a sufficiency condition given by Wegner [14], which we now describe. Recall that a *meet semilattice* is a poset for which any two elements have a unique greatest lower bound. A ranked poset P, with rank function ρ, has the *diamond property* if on any interval $[a, b]$ with $\rho(b) - \rho(a) = 2$, there are at least two distinct elements c and d such that $a < c < b$ and $a < d < b$. Wegner's theorem [14] is the following.

Theorem 2.2 (Wegner)**.** *If* P *is a finite meet semilattice with the diamond property and rank generating function* $\sum_{p \in P} t^{\rho(p)} = \sum \gamma_i t^i$*, then* $(\gamma_0, \gamma_1, \ldots)$ *satisfies the Kruskal-Katona inequalities.*

3. Applications of Wegner's theorem

We begin this section by describing a poset structure on the combinatorial objects enumerated by the γ-vector of the type A Coxeter complex, or equivalently, the permutahedron. (For the reader looking for more background on the Coxeter complex itself, we refer to [4, Section 1.15]; for the permutahedron see [15, Example 0.10].)

Recall that a *descent* of a permutation $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$ is a position $i \in [n-1]$ such that $w_i > w_{i+1}$. A *peak* is a position $i \in [2, n-1]$ such that $w_{i-1} < w_i > w_{i+1}$. We let $\mathrm{des}(w)$ denote the number of descents of w, and we let $\mathrm{peak}(w)$ denote the number of peaks. It is well known that the h-polynomial of the type A_{n-1} Coxeter complex is expressed as:

$$h(A_{n-1}; t) = \sum_{w \in \mathfrak{S}_n} t^{\mathrm{des}(w)}.$$

Foata and Schützenberger were the first to demonstrate the γ-nonnegativity of this polynomial (better known as the *Eulerian polynomial*), showing $h(A_{n-1}; t) = \sum \gamma_i t^i (1+t)^{n-1-2i}$, where γ_i = the number of equivalence classes of permutations of n with $i+1$ peaks [2]. (Two permutations are in the same equivalence class if they have the same sequence of values at their peaks and valleys.) See also Shapiro, Woan, and Getu [10] and, in a broader context, Brändén [1] and Stembridge [13].

Following Postnikov, Reiner, and Williams [9], we choose the following set of representatives for these classes:

$$\widehat{\mathfrak{S}}_n = \{w \in \mathfrak{S}_n : w_{n-1} < w_n, \text{ and if } w_{i-1} > w_i \text{ then } w_i < w_{i+1}\}.$$

In other words, $\widehat{\mathfrak{S}}_n$ is the set of permutations w with no double descents and no final descent, or those for which $\operatorname{des}(w) = \operatorname{peak}(0w0) - 1$. We now phrase the γ-nonnegativity of the type A Coxeter complex in this language.

Theorem 3.1 (Foata-Schützenberger)**.** *The h-polynomial of the type A_{n-1} Coxeter complex can be expressed as follows:*

$$h(A_{n-1};t) = \sum_{w \in \widehat{\mathfrak{S}}_n} t^{\operatorname{des}(w)}(1+t)^{n-1-2\operatorname{des}(w)}.$$

We now can state precisely that the type A_{n-1} Coxeter complex has $\gamma(A_{n-1}) = (\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor \frac{n-1}{2} \rfloor})$, where

$$\gamma_i(A_{n-1}) = |\{w \in \widehat{\mathfrak{S}}_n : \operatorname{des}(w) = i\}|.$$

We impose a partial order $\leq$ on $\widehat{\mathfrak{S}}_n$ so that it is ranked by descent number. We will then show that $\widehat{\mathfrak{S}}_n$ is a meet semilattice with the diamond property. Hence Wegner's theorem applies and we will conclude that $\gamma(A_{n-1})$ satisfies the Kruskal-Katona inequalities.

To more easily see the poset structure, we insert bars just after the descents in elements of $w \in \widehat{\mathfrak{S}}_n$. For example, we write

$$w = 1369|25|478.$$

Thus we can think of these permutations as ordered lists of maximal increasing runs. Notice that the definition of $\widehat{\mathfrak{S}}_n$ means we can have no blocks of size 1 except possibly the first one. We order $\widehat{\mathfrak{S}}_n$ by refinement of blocks. Thus, v covers u if we can obtain u from v by removing a bar between two blocks and reordering the numbers within. For example, $1369|25|478$ covers precisely two other elements: $123569|478$ and $1369|24578$.

The following theorem along with Theorem 2.2 implies part (a) of Theorem 1.2 for type A Coxeter complexes.

Theorem 3.2. *The poset $(\widehat{\mathfrak{S}}_n, \leq)$ is a meet semilattice with the diamond property.*

Corollary 3.3. *The γ-vector of the type A_{n-1} Coxeter complex (permutahedron) satisfies the Kruskal-Katona inequalities.*

Proof of Theorem 3.2. To see that $\widehat{\mathfrak{S}}_n$ is a meet semilattice, consider two permutations in $\widehat{\mathfrak{S}}_n$,

$$w = w_1|w_2|\cdots|w_k \quad \text{and} \quad v = v_1|v_2|\cdots|v_l.$$

Their greatest lower bound u is easily computed as follows. Let i_w (resp. i_v) be the position of leftmost bar in w (resp. v) where $\bigcup_{1\leq j\leq i_w} w_j = \bigcup_{1\leq j\leq i_v} v_j$. Let u_1 be this common set, and proceed by induction on

$w' = w_{i_w+1}|\cdots|w_k$ and $v' = v_{i_v+1}|\cdots|v_l$ to form u_2, and so on. As an example with $n = 9$ let $w = 3|1247|68|59$ and $v = 124|37|58|69$. Then $u = 12347|5689$.

To exhibit the diamond property of $\widehat{\mathfrak{S}}_n$, suppose $u \leq v$ with $[u, v]$ of length two. This means that we obtain u from v by removing exactly two bars from v. But the order in which these bars are removed is unimportant, hence there are two paths down from v to u. □

The permutahedron is an example of a *chordal nestohedron*. Following [9], a chordal nestohedron $P_{\mathcal{B}}$ is characterized by its *building set*, $\mathcal{B}$. Each building set $\mathcal{B}$ on $[n]$ has associated to it a set of *$\mathcal{B}$-permutations*, $\mathfrak{S}_n(\mathcal{B}) \subset \mathfrak{S}_n$, and we similarly define $\widehat{\mathfrak{S}}_n(\mathcal{B}) = \mathfrak{S}_n(\mathcal{B}) \cap \widehat{\mathfrak{S}}_n$. See [9] for details. The following is a main result of Postnikov, Reiner, and Williams [9].

Theorem 3.4 (Postnikov, Reiner, Williams). [9, Theorem 11.6] *If $\mathcal{B}$ is a connected chordal building set on $[n]$, then*

$$h(P_{\mathcal{B}}; t) = \sum_{w \in \widehat{\mathfrak{S}}_n(\mathcal{B})} t^{\operatorname{des}(w)}(1+t)^{n-1-2\operatorname{des}(w)}.$$

Recall that Q is a *lower ideal* of a poset P if: 1) Q is a subposet of P and 2) if $y \in Q$ and $y > x \in P$, then $x \in Q$. It is obvious that any lower ideal of a meet semilattice with the diamond property is itself a meet semilattice with the diamond property. Thus, we can generalize Theorem 3.2 in the following manner.

Lemma 3.5. *If $\mathcal{B}$ is a connected chordal building set on $[n]$ such that $\widehat{\mathfrak{S}}_n(\mathcal{B})$ is a lower ideal in $\widehat{\mathfrak{S}}_n$, then the γ-vector of the chordal nestohedron $P_{\mathcal{B}}$ satisfies the Kruskal-Katona inequalities.*

As a key application of Lemma 3.5, consider the *associahedron*, Assoc_n. This is a well studied object, whose h-vector is given by the *Narayana numbers*. The Narayana numbers in turn have a combinatorial interpretation given by noncrossing partitions. Simion and Ullmann [11] give a particular decomposition of the lattice of noncrossing partitions that may be used to describe $\gamma(\mathrm{Assoc}_n)$, and show that it is Kruskal-Katona. Here we prefer to follow [9, Section 10.2], and observe that the associahedron is a chordal nestohedron whose $\mathcal{B}$-permutations are precisely the 312-avoiding permutations. Let $\mathfrak{S}_n(312)$ denote the set of all $w \in \mathfrak{S}_n$ such that there is no triple $i < j < k$ with $w_j < w_k < w_i$. By Theorem 3.4, we have

$$h(\mathrm{Assoc}_n; t) = \sum_{w \in \widehat{\mathfrak{S}}_n(312)} t^{\operatorname{des}(w)}(1+t)^{n-1-2\operatorname{des}(w)},$$

where $\widehat{\mathfrak{S}}_n(312) = \mathfrak{S}_n(312) \cap \widehat{\mathfrak{S}}_n$. We know that $\widehat{\mathfrak{S}}_n(312)$ is a lower ideal in $\widehat{\mathfrak{S}}_n$, for if $w \in \mathfrak{S}(312)$ is 312-avoiding, clearly any coarsening of w is also 312-avoiding. Thus we can conclude part (b) of Theorem 1.2.

Corollary 3.6. *The γ-vector of the associahedron satisfies the Kruskal-Katona inequalities.*

Remark 3.7. There are many examples of connected chordal building sets $\mathcal{B}$ for which the hypotheses of Lemma 3.5 do not apply, and yet we still believe that the γ-vectors of the related nestohedra satisfy the Kruskal-Katona inequalities. For example, let $\mathcal{B}$ be the building set induced by the graph $(V,E) = ([5], \{13, 14, 23, 25, 34, 35, 45\})$. Then $4|13|25 \in \widehat{\mathfrak{S}}_n(\mathcal{B})$, but $4|1235$ is not. Hence $\widehat{\mathfrak{S}}_n(\mathcal{B})$ is not a lower ideal in $\widehat{\mathfrak{S}}_n$. However, its γ-vector is $(1, 17, 10)$, which satisfies the Kruskal-Katona inequalities.

We now turn our attention to the type B Coxeter complex. The framework of [9] no longer applies, so we must discuss a new, if similar, combinatorial model.

In type B_n, the γ-vector is given by $\gamma_i = 4^i$ times the number of permutations w of $\mathfrak{S}_n$ such that $\operatorname{peak}(0w) = i$. See Petersen [8] and Stembridge [13]. We define the set of *decorated permutations* Dec_n as follows. A decorated permutation $\mathbf{w} \in \operatorname{Dec}_n$ is a permutation $w \in \mathfrak{S}_n$ with bars following the peak positions (with $w_0 = 0$, $w_{n+1} = n+1$). Moreover these bars come in four colors: $\{| = |^0, |^1, |^2, |^3\}$. Thus for each $w \in \mathfrak{S}_n$ we have $4^{\operatorname{peak}(0w)}$ decorated permutations in Dec_n. For example, Dec_n includes elements such as

$$4|238|^1 76519, \quad 4|^3 238|^2 76519, \quad 25|137|^1 69|^2 84.$$

Let $\operatorname{peak}(\mathbf{w}) = \operatorname{peak}(0w)$ denote the number of bars in $\mathbf{w}$. In this context we have the following result.

Theorem 3.8 (Petersen)**.** [8, Proposition 4.15] *The h-polynomial of the type B_n Coxeter complex can be expressed as follows:*

$$h(B_n; t) = \sum_{\mathbf{w} \in \operatorname{Dec}_n} t^{\operatorname{peak}(\mathbf{w})} (1+t)^{n-2\operatorname{peak}(\mathbf{w})}.$$

To give Dec_n the proper poset structure, we let $\leq$ be the order given by refinement, but with the caveat that a block cannot be refined if it is not in increasing order. So, for example, $\mathbf{w} = 34|^1 2156$ is maximal in Dec_n, whereas the second block of $\mathbf{v} = 34|^1 1256$ can be refined further, e.g., $\mathbf{v} < 34|^1 16|52$. The following theorem implies part (a) of Theorem 1.2 for type B Coxeter complexes.

Theorem 3.9. *The poset $(\operatorname{Dec}_n, \leq)$ is a meet semilattice with the diamond property.*

Corollary 3.10. *The γ-vector of the type B_n Coxeter complex satisfies the Kruskal-Katona inequalities.*

Proof of Theorem 3.9. The arguments here are very similar to the case of $\widehat{\mathfrak{S}}_n$. The diamond property is clear. Computing the meet requires only slightly more care.

The meet of two decorated permutations $\mathbf{w} = w_1|^{c_1} w_2|^{c_2} \cdots |^{c_{k-1}} w_k$ and $\mathbf{v} = v_1|^{d_1} v_2|^{d_2} \cdots |^{d_{l-1}} v_l$, is the decorated permutation $\mathbf{u} = u_1|^{e_1} u_2|^{e_2} \cdots$ computed as follows. Let i_w (resp. i_v) be the position of leftmost bar in $\mathbf{w}$ (resp. $\mathbf{v}$) where both

- $\bigcup_{1\leq j\leq i_w} w_j = \bigcup_{1\leq j\leq i_v} v_j$, and
- the colors of the bars agree: $c_{i_w} = d_{i_v}$.

Let $u_1 = \bigcup_{1\leq j\leq i_w} w_j$ as a set, $e_1 = c_{i_w}$. We write u_1 as a sequence in strictly increasing order unless $w_1 = v_1$ *as words* and $i_w = i_v = 1$, in which case u_1 is this word. After u_1 and e_1 are computed, proceed by induction on $\mathbf{w}' = w_{i_w+1}|^{c_{i_w+1}} \cdots |^{c_{k-1}} w_k$ and $\mathbf{v}' = v_{i_v+1}|^{d_{i_v+1}} \cdots |^{d_{l-1}} v_l$ to determine u_2, e_2 and so on. As an example with $n = 9$ let $w = 13|^2 247|^1 689|5$ and $v = 13|247|^1 569|^3 8$. Then $u = 12347|^1 5689$. □

The approach of Theorem 3.9 admits an obvious generalization to any number of colors, though we have no examples of simplicial complexes whose γ-vectors would be modeled by such decorated permutations. Also, notice that $\widehat{\mathfrak{S}}_n$ is a lower ideal in Dec_n, and thus Theorem 3.2 follows from Theorem 3.9.

We now describe how to view the elements enumerated by the γ-vector of the type D Coxeter complex in terms of a subset of decorated permutations. Define a subset $\mathrm{Dec}_n^D \subseteq \mathrm{Dec}_n$ as follows:

$$\mathrm{Dec}_n^D = \{\mathbf{w} = w_1 \cdots |^{c_1} w_{i_1} \cdots |^{c_2} \cdots \in \mathrm{Dec}_n \text{ such that } w_1 < w_2 < w_3, \text{ or,} \\ \max\{w_1, w_2, w_3\} \neq w_3 \text{ and } c_1 \in \{0,1\}\}.$$

In other words, we remove from Dec_n all elements whose underlying permutations have $w_2 < w_1 < w_3$, then for what remains we dictate that bars in the first or second positions can only come in one of two colors. Stembridge [13] gives an expression for the h-polynomial of the type D Coxeter complex, which we now phrase in the following manner.

Theorem 3.11 (Stembridge)**.** [13, Corollary A.5]. *The h-polynomial of the type D_n Coxeter complex can be expressed as follows:*

$$h(D_n;t) = \sum_{\mathbf{w}\in \mathrm{Dec}_n^D} t^{\mathrm{peak}(\mathbf{w})}(1+t)^{n-2\,\mathrm{peak}(\mathbf{w})}.$$

It is easy to verify that the induced subposet $(\mathrm{Dec}_n^D, \leq) \subseteq (\mathrm{Dec}_n, \leq)$ is a lower ideal (when coarsening an element of Dec_n^D, we never result in an element with $w_2 < w_1 < w_3$), and so we obtain the following corollary of Theorem 3.9. This corollary, along with explicit verification of the Kruskal-Katona inequalities for the γ-vectors in Table 1, completes the proof of part (a) of Theorem 1.2.

Corollary 3.12. *The γ-vector of the type D_n Coxeter complex satisfies the Kruskal-Katona inequalities.*

W	$\gamma(W)$
E_6	$(1, 1266, 7104, 3104)$
E_7	$(1, 17628, 221808, 282176)$
E_8	$(1, 881744, 23045856, 63613184, 17111296)$
F_4	$(1, 232, 208)$
G_2	$(1, 8)$
H_3	$(1, 56)$
H_4	$(1, 2632, 3856)$
$I_2(m)$	$(1, 2m - 4)$

TABLE 1. The γ-vectors for finite Coxeter complexes of exceptional type

The cyclohedron Cyc_n, or type B associahedron, is a nestohedron, though not a chordal nestohedron. Its γ-vector can be explicitly computed from its h-vector as described in [9, Proposition 11.15]. We have $\gamma_i(\mathrm{Cyc}_n) = \binom{n}{i,i,n-2i}$. Define

$$P_n = \{(L, R) \subseteq [n] \times [n] : |L| = |R|, L \cap R = \emptyset\}.$$

For $\sigma \in P_n$, let $\rho(\sigma) = |L| = |R|$. Then we can write

$$h(\mathrm{Cyc}_n; t) = \sum_{\sigma \in P_n} t^{\rho(\sigma)}(1 + t)^{n-2\rho(\sigma)}.$$

Define a partial order on P_n as follows. For $\sigma = (L, R)$ with $\rho(\sigma) = k$, write σ as a $k \times 2$ array with the elements of L written in increasing order in the first column, the elements of R in increasing order in the second column. That is, if $L = \{l_1 < \cdots < l_k\}$ and $R = \{r_1 < \cdots < r_k\}$, we write

$$\sigma = \begin{pmatrix} l_1 & r_1 \\ \vdots & \vdots \\ l_k & r_k \end{pmatrix}.$$

We define $\sigma \leq \tau$ if and only if the set of rows of σ (i.e., the set of pairs (l_i, r_i)) is contained in the set of rows of τ. The following theorem gives part (c) of Theorem 1.2, completing its proof.

Theorem 3.13. *The poset $(P_n, \leq)$ is a meet semilattice with the diamond property.*

Corollary 3.14. *The γ-vector of the cyclohedron satisfies the Kruskal-Katona inequalities.*

Proof of Theorem 3.13. As the partial order is simply containment order, we see that the diamond property is clear (every interval of length two contains precisely four elements) and the meet of two elements is given by the intersection of their row-sets. □

4. Flag spheres with few vertices

We now describe a different class of flag spheres whose γ-vectors satisfy the Kruskal-Katona inequalities: those with few vertices relative to their dimension. Our starting point is the following lemma, see [6] and [3, Lemma 2.1.14]. (Recall that the boundary of the d-dimensional *cross-polytope* is the d-fold join of the zero-dimensional sphere, called also *octahedral sphere*.)

Lemma 4.1 (Meshulam, Gal)**.** *If Δ is a flag homology sphere then:*

(a) $\gamma_1(\Delta) \geq 0$,
(b) *if* $\gamma_1(\Delta) = 0$*, then Δ is an octahedral sphere.*

By definition, if Δ is a $(d-1)$-dimensional flag homology sphere, we have $f_1(\Delta) = 2d + \gamma_1(\Delta)$. For Theorem 1.3 we will classify γ-vectors of those Δ for which $0 \leq \gamma_1(\Delta) \leq 2$, or equivalently $2d \leq f_1(\Delta) \leq 2d+2$. Notice that an octahedral sphere (of any dimension) has $\gamma = (1, 0, 0, \ldots)$, and so we only need to consider the cases $\gamma_1 = 1$ and $\gamma_1 = 2$.

If Δ is a flag homology d-sphere, $F \in \Delta$ and $|F| = k$, then $\mathrm{lk}(F)$ is a flag homology $(d-k)$-sphere (for flagness see Lemma 4.2(b) below). The *contraction* of the edge $\{u, v\}$ in Δ is the complex $\Delta' = \{F \in \Delta : u \notin F\} \cup \{(F \setminus \{u\}) \cup \{v\} : F \in \Delta, u \in F\}$. By [7, Theorem 1.4] Δ' is a sphere if Δ is a sphere, but it is not necessarily flag. We have the following relation of γ-polynomials:

$$\gamma(\Delta; t) = \gamma(\Delta'; t) + t\gamma(\mathrm{lk}(\{u, v\}); t). \tag{1}$$

Also, the *suspension* $\mathrm{susp}(\Delta) = \Delta \cup \{\{a\} \cup F, \{b\} \cup F : F \in \Delta\}$ (for vertices a and b not in the vertex set of Δ), of a flag sphere Δ has the same γ-polynomial as Δ:

$$\gamma(\mathrm{susp}(\Delta); t) = \gamma(\Delta; t).$$

In the following lemma we collect some known facts and some simple observations which will be used frequently in what follows.

Lemma 4.2. *Let Δ be a flag complex on vertex set V. Then the following holds:*

(a) If $A \subseteq V$ then $\Delta[A]$ is flag.
(b) If $F \in \Delta$ then $\mathrm{lk}(F)$ is an induced subcomplex of Δ, hence flag.
(c) Let Δ' be obtained from Δ by contracting an edge $\{u, v\}$. If Δ' is not flag then $\{u, v\}$ is contained in an induced 4-cycle in Δ.

Let K be a simplicial complex on vertex set U and Γ a subcomplex of K on vertex set A. Then:

(d) If $\Gamma = K[A]$ then $K - \Gamma$ deformation retracts on $K[U - A]$.
(e) If K and Γ are spheres then $K - \Gamma$ is homological to a sphere of dimension $\dim K - \dim \Gamma - 1$. In particular, if $\dim K = \dim \Gamma$ then $K = \Gamma$.

Proof. Part (a) is obvious. For (b), Let v be a vertex of Δ. If all proper subsets of a face $T \in \Delta$ are in $\mathrm{lk}(v)$, then by flagness $T \cup \{v\} \in \Delta$, hence $T \in \mathrm{lk}(v)$ so $\mathrm{lk}(v)$ is an induced subcomplex. If $F = T \cup \{v\}$ in Δ where

$v \notin T$, then $\mathrm{lk}_\Delta(F) = \mathrm{lk}_{\mathrm{lk}(v)}(T)$, and by induction on the number of vertices in F we conclude that $\mathrm{lk}(F)$ is an induced subcomplex. By part (a) it is flag, concluding (b).

To prove (c), suppose Δ' is not flag and let F be a minimal set such that all proper subsets of F are in Δ' but $F \notin \Delta'$. Then by flagness of Δ there must exist two vertices $v' \neq u'$ in F such that $\{v, v'\} \notin \Delta$ and $\{u, u'\} \notin \Delta$. Then (v, u, v', u') is an induced 4-cycle in Δ, proving (c).

Part (d) is easy and well known, and (e) is a consequence of Alexander duality. □

Proposition 4.3. *If Δ is a $(d-1)$-dimensional flag sphere with $\gamma_1(\Delta) = 1$, then $\gamma(\Delta; t) = 1 + t$.*

Proof. We will proceed by induction on dimension. As a base case $d = 2$, observe that for Δ the boundary of an n-gon one has $f(\Delta; t) = 1 + nt + nt^2$, $h(\Delta; t) = 1 + (n-2)t + t^2$, and hence $\gamma(\Delta; t) = 1 + (n-4)t$ and $\gamma_1(\Delta) = 1$ only for the pentagon.

Now suppose Δ is a $(d-1)$-dimensional flag sphere with $2d+1$ vertices. If Δ is a suspension, it is a suspension of a $(d-2)$ sphere with $2d - 1 = 2(d-1) + 1$ vertices and we are finished by induction. Otherwise, the link of any vertex v is a $(d-2)$-sphere with precisely $2d - 2$ vertices, i.e., an octahedral sphere.

Consider in the latter case a vertex a and the two vertices b, c in the interior of its antistar. If $\{b, c\}$ is not an edge in Δ, the $(d-2)$-sphere $\mathrm{lk}(b)$ must be contained in the induced subcomplex $\mathrm{lk}(a)$ and by Alexander duality we get $\mathrm{lk}(b) = \mathrm{lk}(a)$. Deleting c gives a proper subcomplex of Δ that is itself a $(d-1)$-sphere (the suspension over $\mathrm{lk}(a)$), an impossibility. Thus we conclude $\{b, c\}$ must be an edge in Δ. Now if we contract $\{b, c\}$ to obtain Δ', we see that Δ' is nothing but the suspension of $\mathrm{lk}(a)$ by a and c. As $\mathrm{lk}(a)$ is an octahedral sphere, we know $\gamma(\mathrm{lk}(a); t) = \gamma(\Delta'; t) = 1$.

Now we show that $\mathrm{lk}(\{b, c\})$ is an octahedral sphere as well. Note that the $(d-3)$-sphere $\mathrm{lk}(\{b, c\})$ is a subcomplex of codimension 1 in the octahedral sphere $\mathrm{lk}(a)$. Thus, $\mathrm{lk}(\{b, c\})$ misses at least one pair of antipodal vertices in $\mathrm{lk}(a)$, and by Alexander duality $\mathrm{lk}(\{b, c\})$ is an octahedral sphere of the form $\mathrm{lk}(a)[U - \{u, u'\}]$ where U is the vertex set of $\mathrm{lk}(a)$ and $\{u, u'\} \subseteq U$ is a pair of antipodal vertices.

Thus we conclude from Equation (1) that

$$\gamma(\Delta; t) = \gamma(\Delta') + t\gamma(\mathrm{lk}(\{b, c\}); t) = 1 + t.$$

□

Proposition 4.4. *If Δ is a $(d-1)$-dimensional flag sphere with $\gamma_1(\Delta) = 2$, then $\gamma(\Delta; t) \in \{1 + 2t, 1 + 2t + t^2\}$.*

Proof. Again we proceed by induction on dimension. For base case $d = 2$, as we observed beforehand $\gamma_1(\Delta) = 2$ only for the boundary of a hexagon, in which case $\gamma(\Delta; t) = 1 + 2t$. Assume $d > 2$.

We now analyze the structure of Δ according to the number of vertices in the interior of the antistar of a vertex $v \in \Delta$, denoted by $i(v)$. Always $i(v) > 0$ as Δ is flag with nontrivial top homology (use Lemma 4.2(b) with $F = \{v\}$).

If there is a vertex $v \in \Delta$ with $i(v) = 1$, then Δ is the suspension over $\mathrm{lk}(v)$, and we are done by induction on dimension.

If there is a vertex $v \in \Delta$ with $i(v) = 2$, let b and c denote the vertices in the interior of its antistar. We conclude as in the proof of Proposition 4.3 that $\{b, c\} \in \Delta$. Let Δ' be obtained from Δ by contracting the edge $\{b, c\}$. Then Δ' is also a flag sphere (it is the suspension over $\mathrm{lk}(v)$), $\gamma(\Delta') = 1+t$ by Proposition 4.3, and $\gamma_1(\mathrm{lk}(\{b, c\})) \in \{0, 1, 2\}$ by the induction hypothesis. We now show that $\gamma_1(\mathrm{lk}(\{b, c\})) = 2$ is impossible.

Assume by contradiction that $\gamma_1(\mathrm{lk}(\{b, c\})) = 2$. Then there are exactly 4 vertices in Δ which are not in $\mathrm{lk}(\{b, c\})$, and they include v, b, c. Call the fourth vertex e. By Lemma 4.2(b) $\mathrm{lk}(\{b, c\})$ is an induced subcomplex of codimension 1 in $\mathrm{lk}(v)$, and by 4.2(e) $\mathrm{lk}(v) - \mathrm{lk}(\{b, c\})$ is homologous to the zero dimensional sphere. On the other hand, by 4.2(d) $\mathrm{lk}(v) - \mathrm{lk}(\{b, c\})$ deformation retracts onto e in $\mathrm{lk}(v)$, a contradiction. Thus $\gamma(\mathrm{lk}(\{b, c\}) \in \{1, 1 + t\}$. By (1), $\gamma(\Delta) \in \{1 + 2t, 1 + 2t + t^2\}$ in this case.

The last case to consider is when $i(v) = 3$ for every vertex $v \in \Delta$. In this case, any $\mathrm{lk}(v)$ is an octahedral sphere. Let $a \in \Delta$ and $I(a) = \{b, c, e\}$ be the set of interior vertices in $\mathrm{ast}(a)$.

Suppose there exists an edge in Δ with both ends in $I(a)$ such that its contraction results in a flag complex. Now we can contract an edge, say $\{b, c\}$, so that the result is a flag complex Δ'. By [7, Theorem 1.4] Δ' is a sphere, and by Proposition 4.3 $\gamma(\Delta') = 1 + t$. As $\mathrm{lk}\{b, c\}$ is octahedral, by (1) we get $\gamma(\Delta) = \gamma(\Delta') + t = 1 + 2t$.

To conclude the proof, we will show there always exists such an edge. First we see that $\Delta[b, c, e]$ can appear in only one of two forms. By 4.2(d) and (e) the induced subcomplex $\Delta[a, b, c, e]$ is homotopic to $\Delta - \mathrm{lk}(a)$ and hence homologous to the zero dimensional sphere. Thus, as $\Delta[b, c, e]$ is flag, it is either a triangle or a 3-path, say (b, c, e). To obtain a contradiction, suppose the contraction of no edge contained in $I(a)$ yields a flag complex. Then by 4.2(c) any edge contained in $I(a)$ belongs to some induced 4-cycle C in Δ. First we show that C cannot contain $I(a)$: if this is the case then $\Delta[b, c, e]$ is a 3-path and $C = (b, c, e, x)$ for some $x \in \mathrm{lk}(a)$. As $\mathrm{lk}(a)$ is an octahedral sphere, the interior of the antistar of x in Δ contains only 2 vertices (c and one vertex from $\mathrm{lk}(a)$), a contradiction as $i(x) = 3$. Now assume without loss of generality that $C = (b, c, x, y)$ with $x, y \in \mathrm{lk}(a)$. By the argument above, e belongs to the interior of the antistars both of x and y. On the other hand, the edge $\{x, y\}$ is contained in a facet with a vertex from the interior of the antistar of a, a contradiction. □

Note that if Δ is the join of the boundaries of two pentagons then $\gamma(\Delta; t) = 1 + 2t + t^2$.

This completes the proof of Theorem 1.3.

References

[1] P. Brändén, Sign-graded posets, unimodality of W-polynomials and the Charney-Davis conjecture, *Electron. J. Combin.* **11** (2004/06), Research Paper 9, 15pp.

[2] D. Foata and M.-P. Schützenberger, "Théorie géométrique des polynômes eulériens," Lecture Notes in Mathematics, Vol. 138, Springer-Verlag, Berlin, 1970.

[3] S. R. Gal, Real root conjecture fails for five- and higher-dimensional spheres, *Discrete Comput. Geom.* **34** (2005), 269–284.

[4] J. E. Humphreys, "Reflection groups and Coxeter groups," Cambridge Univ. Press, Cambridge, 1990.

[5] K. Karu, The *cd*-index of fans and posets, *Compos. Math.* **142** (2006), 701–718.

[6] R. Meshulam, Domination numbers and homology, *J. Combin. Theory Ser. A* **102** (2003), 321–330.

[7] E. Nevo, Higher minors and Van Kampen's obstruction, *Math. Scand.* **101** (2007), 161–176.

[8] T. K. Petersen, Enriched P-partitions and peak algebras, *Adv. Math.* **209** (2007), 561–610.

[9] A. Postnikov, V. Reiner, and L. Williams, Faces of generalized permutohedra, Doc. Math. 13 (2008) 207–273.

[10] L. W. Shapiro, W. J. Woan, S. Getu, Runs, slides and moments, *SIAM J. Algebraic Discrete Methods* **4** (1983), 459–466.

[11] R. Simion and D. Ullman, On the structure of the lattice of noncrossing partitions, *Discrete Math.* **98** (1991), 193–206.

[12] R. P. Stanley, "Combinatorics and Commutative Algebra" (2nd ed.), Birkhäuser, Boston, 1996.

[13] J. R. Stembridge, Coxeter cones and their h-vectors, Adv. Math. 217 (2008) 1935–1961.

[14] G. Wegner, Kruskal–Katona's theorem in generalized complexes. "Finite and infinite sets, Vol. I, II" (Eger, 1981), 821–827, Colloq. Math. Soc. Jnos Bolyai, 37, North-Holland, Amsterdam, 1984.

[15] G. M. Ziegler, "Lectures on polytopes," Graduate Texts in Mathematics, **152**, Springer-Verlag, New York, 1995.

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