LINEAR RESOLUTIONS of POWERS of EDGE IDEALS

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Abstract: We discuss the linearity of the minimal free resolution of a power of an edge ideal.

1. Introduction

Throughout, $S = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k, and G is a finite simple graph (that is, without loops and multiple edges) on vertex set $\{x_1, \ldots, x_n\}$. The monomial *edge ideal* associated to G is

$$I_G = (\, x_i x_j \mid x_i x_j \in G \,)\,,$$

where the edge $\{x_i, x_j\}$ is denoted $x_i x_j$ for short. The homological properties of I_G depend on the combinatorial properties of G and of the *complement graph* G^c with edges $\{x_i x_j \mid x_i x_j \notin G\}$. For example, the following definitions are helpful: we say that a simple graph T contains a *q-cycle* if there exist distinct vertices $x_{i_1}, \ldots, x_{i_q}$ such that $x_{i_q} x_{i_1} \in T$ and $x_{i_j} x_{i_{j+1}} \in T$ for all $1 \leq j \leq q-1$; a *chord* in the cycle is an edge in T between two non-consecutive vertices; a cycle is called *induced* if it has no chords.

By polarization, studying the minimal free resolutions of quadratic monomial ideals is equivalent to studying the minimal free resolutions of edge ideals. Describing all such resolutions is beyond reach since they can have very complicated structure. In fact, very little is known about their numerical invariants: the graded Betti numbers $\beta_{i,j}(I_G)$ and regularity $\operatorname{reg}(I_G) = \max\{j - i \mid \beta_{i,j}(I_G) \neq 0\}$. The following problem is wide open: find upper and lower bounds on $\operatorname{reg}(I_G)$ in terms of the combinatorial properties of the graphs G and G^c. Even less is known about the minimal free resolutions of powers of edge ideals.

The simplest case is when the regularity is as minimal as possible, that is, when $\operatorname{reg}(I_G^s) = 2s$; in this case, we say that the *minimal free resolution is linear.* The following is known:

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Theorem 1.1. [Fr] *The minimal free resolution of I_G is linear if and only if the complement graph G^c is chordal (that is, every induced cycle in G^c is a triangle).*

Theorem 1.2. [HHZ] *If G^c is chordal, then for every $s \geq 2$ the minimal free resolution of I_G^s is linear.*

The next natural question is:

Question 1.3. *For what combinatorial properties of G (and G^c) and for what $s \geq 2$ is the minimal free resolution of I_G^s linear?*

Based on many Macaulay2 examples, computed by C. Francisco, the following possibility seemed reasonable:

Question 1.4. (Francisco-Hà-Van Tuyl; personal communication) *Is it true that I_G^s has a linear resolution for all $s \geq 2$ if and only if G^c has no induced 4-cycles?*

We say that the edges x_ix_j and x_px_q in G are *disjoint* if $x_ix_px_jx_q$ is a 4-cycle in G^c. The combinatorial condition that G^c has no induced 4-cycles has the following algebraic meaning:

Proposition 1.5. (Francisco-Hà-Van Tuyl; non-published) *G^c has no induced 4-cycles if and only if the Betti numbers $\beta_{1,j}(I_G)$ vanish for $j > 3$ (that is, I_G has only linear minimal first syzygies).*

Proof: By Taylor's resolution, it follows that $\beta_{1,j}(I_G) = 0$ for $j \neq 3, 4$. By [HV, Theorem 3.2.4], $\beta_{1,4}(I_G) = 0$ if and only if G^c has no induced 4-cycles. □

Francisco, Hà, and Van Tuyl proved the following result, which provides one direction of 1.4. We present our own short proof.

Proposition 1.6. (Francisco-Hà-Van Tuyl; non-published) *If I_G^s has a linear resolution for some $s \geq 1$, then G^c has no induced 4-cycles.*

Proof: Suppose that there exist two disjoint edges x_ix_j and x_px_q in G. By [GPW], the Betti numbers of I_G^s can be computed using the lcm-lattice $L(I_G^s)$ of I_G^s. The monomials $(x_ix_j)^s$ and $(x_px_q)^s$ are atoms in the lattice. The monomial $(x_ix_j)^s(x_px_q)^s$ covers these two atoms since the edges are disjoint. Therefore, the open interval $\big(1, (x_ix_j)^s(x_px_q)^s\big)$ consists of the two atoms. Hence, $\beta_{1,(x_ix_j)^s(x_px_q)^s} = \tilde{H}_0\Bigg(\big(1, (x_ix_j)^s(x_px_q)^s\big); k\Bigg) = 1$. Since $\deg((x_ix_j)^s(x_px_q)^s) = 4s$, we get that the graded Betti number $\beta_{1,4s} \neq 0$, so the minimal free resolution of I_G^s is not linear. □

Thus, Question 1.4 is reduced to the following:

Question 1.7. *Is it true that if G^c has no induced 4-cycles then I_G^s has a linear resolution for all $s \geq 2$?*

We present a counterexample:

Counterexample 1.8. We felt that candidates for counterexamples are the graphs such that G^c has no induced 4-cycles and the clique complex of G^c is a triangulation of a sphere. If the sphere has a high dimension, then the known graphs of this type have many edges and vertices, so the examples cannot be checked with Macaulay2. However, we can construct small such graphs if the sphere is 2-dimensional.

We will define a graph Q. Let Q have vertices $a_1, a_2, a_3, a_4, a_5, a_6$, and $b_1, b_2, b_3, b_4, b_5, b_6$. Let the edges of Q^c be the edges of an icosahedron on these vertices. For those who would like to verify our computer computations, we list the edges of Q^c:

$$\begin{aligned}
&a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1 \text{(from the bottom pentagon)}\\
&b_1b_2, b_2b_3, b_3b_4, b_4b_5, b_5b_1 \text{(from the top pentagon)}\\
&a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_1b_2, a_2b_3, a_3b_4, a_4b_5, a_5b_1 \text{(from the walls)}\\
&a_6a_1, a_6a_2, a_6a_3, a_6a_4, a_6a_5 \text{(from the bottom cone)}\\
&b_6b_1, b_6b_2, b_6b_3, b_6b_4, b_6b_5 \text{(from the top cone)}\,.
\end{aligned}$$

In particular, Q^c has 12 vertices and 30 edges. The graph Q^c contains no induced 4-cycles and its clique complex is a triangulation of a 2-sphere. Thus, the graph Q is of the desired type. Computation with Macaulay2 shows that

$$\operatorname{reg}(I_Q) = 4 \quad \text{and} \quad \operatorname{reg}(I_Q^2) = 5\,.$$

In view of 1.8, we have to modify Question 1.4 as follows.

Question 1.9. *Is it true that I_G^s has a linear resolution for all $s \geq 2$ if G^c has no induced 4-cycles and $\operatorname{reg}(I_G) \leq 3$?*

The Macaulay2 examples, computed by C. Francisco, provide evidence that 1.9 has a positive answer. Nevo [Ne] proves that if G is claw-free (in particular, in the case when G^c is a q-cycle and $q \geq 5$), then I_G^2 has a linear resolution.

In view of some special examples (similar to 1.8), we raise the following conjecture.

Conjecture 1.10. *Suppose that G^c has no induced 4-cycles. For $s \geq 1$, we have that*

$$\operatorname{reg}(I_G^{s+1}) \leq \max\{2s+2,\ \operatorname{reg}(I_G^s)+1\}\,.$$

Note that the inequality $2s+2 \leq \operatorname{reg}(I_G^{s+1})$ holds since I_G^{s+1} is generated in degree $2s+2$. If the conjecture holds, then it implies the following answer to Question 1.3.

Conjecture 1.11. *The following conditions are equivalent:*
(1) I_G^s has a linear resolution for some $s \geq 2$.
(2) I_G^s has a linear resolution for every $s \geq \operatorname{reg}(I_G)-1$.
(3) I_G has only linear minimal first syzygies, that is, $\beta_{1,j}(I_G)=0$ for $j>3$.
(4) G^c has no induced 4-cycles.
(5) G has no disjoint edges.

We will show that if 1.10 holds, then 1.11 holds: (4) and (5) are obviously equivalent. (3) and (4) are equivalent by Proposition 1.5. (1) implies (4) by Proposition 1.6. Obviously, (2) implies (1). Finally, (2) follows from (4) by Conjecture 1.10.

The following related question is open: How large can $\operatorname{reg}(I_G)$ be if G^c has no induced 4-cycles?

We remark that Francisco, Hoefel, and Van Tuyl have developed a script for Macaulay2 for computations with edge ideals [FHV].

2. Graded lcm-lattices

Let M be a monomial ideal in the polynomial ring S. Let L_M be the lcm-lattice of M introduced in [GPW]. The atoms of the lattice are the minimal monomial generators of M. The elements in L_M are the least common multiples of the atoms ordered by divisibility; in particular the bottom element is 1, considered as the lcm of the empty set. For an open interval $(1,m)_{L_M}$ in L_M, we denote by $\Delta(1,m)_{L_M}$ the order complex of the interval, and we set $\tilde{H}_{i-1}((1,m)_{L_M};k)=\tilde{H}_{i-1}(\Delta(1,m)_{L_M};k)$ denoting reduced homology with coefficients in k. By [GPW], the Betti numbers of M can be computed by the homology of the open intervals in L_M as follows: if $m \notin L_M$ then $\beta_{i,m}(M)=0$ for every i, and if $m \in L_M$ and $i \geq 1$ we have

$$\beta_{i,m}(M)=\dim_k \tilde{H}_{i-1}((1,m)_{L_M};k)\,. \tag{2.1}$$

By [Ph], if a monomial ideal has a linear minimal free resolution, then its lcm-lattice is graded. Therefore, the next theorem supports Conjecture 1.11. It also shows that the tools for studying graded poset topology are applicable in our case.

Theorem 2.2. *If G^c has no induced 4-cycle, then for any $s \geq 1$ the lcm-lattice $L(I_G^s)$ is graded, and except for the minimum, the rank function is given by* $\mathrm{rank}(m) = \deg(m) - 2s + 1$ *(here, m is a monomial).*

Proof: Let $1 \neq m, m' \in L(I_G^s)$ be monomials such that m divides m' and $\deg(m') - \deg(m) > 1$. We need to show the existence of a monomial $h \in L(I_G^s) \setminus \{m, m'\}$ such that the divisibility conditions $m \,|\, h \,|\, m'$ hold.

There exists a variable x and a non-negative integer r such that x^r divides m, x^{r+1} does not divide m, and x^{r+1} divides m' (take $r = 0$ in case $s = 1$). Let N be the set of neighbors of x in the subgraph induced by G on the set $\mathrm{supp}(m')$. Let $a \in (1, m]$ be an atom, such that x^r divides a. Let $a = \Pi_{1 \leq i \leq s} v_i u_i$, where $v_i u_i \in G$. We consider the following two cases.

Case 1: Suppose that there exists an i such that $v_i \in N$ and $u_i \neq x$. In this case, $b = \dfrac{a}{v_i u_i} \cdot (v_i x)$ is an atom in $(1, m']$, and $h = b \vee m = mx$ is a monomial of the desired type.

Case 2: Suppose that there exists no i such that $v_i \in N$ and $u_i \neq x$. Since x^{r+1} divides m', it follows that x^{r+1} divides some atom, hence $s \geq r + 1$. Therefore, there exists a j such that x does not divide $v_j u_j$. By the assumption in Case 2, it follows that $v_j u_j$ is disjoint from the set $N \cup \{x\}$.

We have that

$$\sum_{w \in N} \deg_{m'}(w) \geq \deg_{m'}(x) \geq r + 1 > r = \sum_{w \in N} \deg_a(w)\,,$$

where $\deg_g(z)$ denotes the exponent of a variable z in a monomial g. Therefore, there exists a q such that $v_q \in N$ and $\deg_a(v_q) < \deg_{m'}(v_q)$.

Since G^c has no induced 4-cycle, there exists an edge e connecting the edges $v_j u_j$ and $v_q x$. This edge must be either $e = yx$ or $e = yv_q$, with $y \in v_j u_j$. If $e = yx$, then $c = \dfrac{a}{v_j u_j} yx$ is an atom in $(1, m']$ and $h = c \vee m = mx$ is a monomial of the desired type. Now, suppose that $e = yv_q$. Consider $b = \dfrac{a}{v_j u_j} \cdot (yv_q)$, which is an atom in $(1, m']$. If b is also an atom in $(1, m]$, then apply Case 1 to the atom b. Otherwise, $h = b \vee m = mv_q$ is a monomial of the desired type. □

For a simplicial complex Γ, let $\alpha(\Gamma)$ denote the largest codimension of a non-vanishing reduced homology, and set $\alpha(\Gamma) = 0$ if Γ is acyclic. For an open interval $(1, m)_{L_M}$, we set $\alpha(1, m)_{L_M} = \alpha\big(\Delta(1, m)_{L_M}\big)$.

Proposition 2.3. *If G^c has no induced 4-cycle, then for any $s \geq 1$ we have*

$$\mathrm{reg}(I_G^s) = 2s + \max_{m \in L_{I_G^s},\, m \neq 1} \left\{ \alpha(1, m)_{L_{I_G^s}} \right\}.$$

Proof: Denote $M = I_G^s$. Applying (2.1) and 2.2, we get

$$\begin{aligned}
\operatorname{reg}(M) &= \max\{j - i \mid \beta_{i,j}(M) \neq 0\} \\
&= \max\{\deg(m) - i \mid \tilde{H}_{i-1}((1,m)_{L_M}; k) \neq 0\} \\
&= 2s - 1 + \max\{\deg(m) - 2s + 1 - i \mid \tilde{H}_{i-1}((1,m)_{L_M}; k) \neq 0\} \\
&= 2s - 1 + \max\{\operatorname{rank}(m) - i \mid \tilde{H}_{i-1}((1,m)_{L_M}; k) \neq 0\} \\
&= 2s + \max\{\operatorname{rank}(m) - 2 - p \mid \tilde{H}_p((1,m)_{L_M}; k) \neq 0\} \\
&= 2s + \max\{\dim \Delta(1,m)_{L_M} - p \mid \tilde{H}_p((1,m)_{L_M}; k) \neq 0\} \\
&= 2s + \max\{q \mid \tilde{H}_{\dim \Delta(1,m)_{L_M} - q}((1,m)_{L_M}; k) \neq 0\} \\
&= 2s + \max_{m \in L_M,\, m \neq 1} \left\{ \alpha(1,m)_{L_M} \right\}.
\end{aligned}$$

□

By 2.3 it follows that Conjecture 1.10 can be expressed in terms of lcm-lattices as follows: if G^c has no induced 4-cycle, then for any $s \geq 1$ we have

$$\max_{m \in L_{I_G^{s+1}},\, m \neq 1} \left\{ \alpha(1,m)_{L_{I_G^{s+1}}}) \right\} \leq \max \left\{ 0, \max_{m \in L_{I_G^s},\, m \neq 1} \left\{ \alpha(1,m)_{L_{I_G^s}}) \right\} - 1 \right\}.$$

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