1. (08/22) Describe the nilpotent elements of $M_{n \times n}(F)$, where $F$ is a field.

2. (08/22) Construct a (commutative) ring $R$ and ideal $I$ of $R$ such that $I \subseteq \mathfrak{m}$ and $I^k \neq 0$ for all $k > 0$.

3. (8/24) Let $F$ be a field. What relationship, if any, is there between the Zariski topology on Spec $F[x_1,\ldots,x_n]$ and the usual Zariski topology on $F^n$?

4. (8/24) For which $y \in \mathbb{Z}_p$ does there exist $x \in \mathbb{Z}_p$ with $x^2 = y$?

5. (8/29) Let $H_0 : \mathbb{N} \to \mathbb{N}$ be eventually equal to a degree $n$ polynomial $f$. Then there exists $h \in \mathbb{Z}[\lambda]$ such that the power series

$$
\sum_{i=0}^{\infty} H(i) \lambda^i = \frac{h(\lambda)}{(1 - \lambda)^n}.
$$

Furthermore, there exist other $H'$ which are also eventually equal to $f$, but whose corresponding $h'$ has arbitrarily high degree.

6. (8/29) Let $Q$ be a finite poset. Define $f_i$ to be the number of chains in $Q$ of length $i$. Let $M = F[x_a : a \in Q]/I_Q$, where as before $I_Q$ is the ideal generated by the monomials $x_a x_b$, $a,b$ not comparable in $Q$. Compute the Hilbert polynomial in terms of the $f_i$.

7. (8/29) What polynomials are the Hilbert polynomials of finitely generated modules over $F[x_1,\ldots,x_n], F$ a field?

8. (8/31) Write down the definition of the left-derived functors of a covariant right-exact functor. Using only the definition, prove that $R^0 \mathfrak{S}(M) = \mathfrak{S}(M)$ for any contravariant left-exact additive functor from $R$-mod to $R$-mod.

9. (9/05) Compute $\text{Ext}^i(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ for all $m,n,i \in \mathbb{N}$. Show that if $M \oplus N$ is a free module, then $\text{Ext}^i(M, L) = \text{Ext}^i(N, L) = 0$ for all $i \geq 1$ and $R$-modules $L$. 

10. (9/07) An extension of $M$ by $N$ is a short exact sequence

$$
0 \to N \to E \to M \to 0.
$$

Two extensions are equivalent if there is a commutative diagram

$$
\begin{array}{cccc}
0 & \to & N & \to & E & \to & M & \to & 0 \\
\downarrow{id} & & \downarrow{\phi} & & \downarrow{id} & & \\
0 & \to & N & \to & E' & \to & M & \to & 0.
\end{array}
$$

(a) If two extensions of $M$ by $N$ are equivalent, then $\phi$ is an isomorphism.

(b) There is a bijection between the set of equivalence classes of extensions of $M$ by $N$ and $\text{Ext}^1(M, N)$.

11. (9/10) We have defined $\text{Ext}^i(M, N)$ as the right-derived functor of the left-exact contravariant functor $\text{Hom}(\ast, N)$. Define $\overline{\text{Ext}}^i$ as the right-derived functor of the left-exact covariant functor $\text{Hom}(M, \ast)$. Prove that $\text{Ext}^i(M, N) \simeq \overline{\text{Ext}}^i(M, N)$.

12. (9/12) Prove the existence of injective resolutions in the category of graded $R$-modules. See Eisenbud, Exerc. A3.5.

13. (9/14)

(a) Prove that $\text{Tor}_i(M, N) \simeq \text{Tor}_i(N, M)$.

(b) Let $r$ be a nonzero divisor of $R$. Compute $\text{Tor}_i(R/(r), M)$.

(c) Let $I, J$ be ideals of $R$. What is $\text{Tor}_1(R/I, R/J)$?

14. (9/17) Let $R = \bigoplus_{i \geq 0} R_i$ be a graded Noetherian ring and $M$ a finitely generated graded $R$-module.

(a) (Graded Nakayama’s Lemma) Let $I = \bigoplus_{i \geq 1}$. If $M \subseteq IM$, then $M = 0$. 

15. (9/22) Let $I$ be an ideal of $R$. Except for part (15a), assume $R$ is Noetherian and $M$ is f.g. $R$-module. Define $\mathfrak{H}_I(M) = \{x \in M : \exists n, I^n x = \{0\}\}$ and given $f : M \to N$ define $\mathfrak{H}_I(f)$ to be the restriction of $f$ to $\mathfrak{H}_I(M)$.

(a) $\mathfrak{H}_I$ is a left-exact covariant functor. If $R$ is a graded ring and $M$ and $N$ are graded submodules, then $\mathfrak{H}_I(M)$ and $\mathfrak{H}_I(N)$ are graded and $\mathfrak{H}_I(f)$ is a graded homomorphism. Denote the right-derived functors of $\mathfrak{H}_I$ by $\mathfrak{H}_I^j(M)$.

(b) A sequence $(r_1, \ldots, r_n)$ of elements of $R$ is called a regular sequence of length $n$ for $M$ if
   - For all $1 \leq i \leq n$ the map $r_i : M/(< r_1, \ldots, r_{i-1} > M) \to M/(< r_1, \ldots, r_{i-1} > M)$ given by $x \to r_i x$ is injective.
   - The quotient module $M/ < r_1, \ldots, r_n > M$ is not zero.

Prove that if $M \neq 0$ and $H^j_I(M) = \cdots = H^0_I(M) = 0$, then there exists a regular sequence of length $n+1$ all of whose elements are in $I$.

[Note: As stated this is not quite true. For instance, $R = \mathbb{Z}, I = 2\mathbb{Z}$ and $M = \mathbb{Z}/5\mathbb{Z}$. While $H^0_I(M) = 0$, no nonzero element of $I$ satisfies the second condition. Can you think of a reasonable restriction on $I$ so that it is true? It should include ideals of local rings and graded ideals of graded rings.]

(c) Suppose that $R = \bigoplus_{i=0}^{\infty} R_i$ is a graded ring, $M$ is a graded module and $I = \bigoplus_{i=1}^{\infty} R_i$ is the irrelevant ideal. Then the Hilbert polynomial of $H^0_I(M)$ is zero.

(d) In addition to the assumptions of the previous problem, assume that $R_0$ is a field. Show that $H^0_I(M) \simeq H^0_I(M/H^0_I(M))$ for all $i \geq 1$.

16. (9/24) Let $R$ be a graded ring and $M$ a graded $R$-module. If $P = \text{ann}(x), x \in M$ is prime, the $P$ is a graded ideal of $R$. What if $P$ is not prime?

17. (10/01) Let $f : R \to S$ be a ring homomorphism such that $f^\ast : \text{Spec} S \to \text{Spec} R$ is a closed map. Show that $f$ satisfies the Cohen-Seidenberg going up lemma. Prove that the converse holds if $S$ is Noetherian. What if $S$ is not Noetherian?

18. (10/03) Graded Noether normalization. Let $R$ be a f.g. $k$-algebra, $k$ a field. Then for any set of generators $x_1, \ldots, x_n$ we can order them so that $x_1, \ldots, x_r$ are algebraically independent over $k$ and $R$ is algebraic over $k[x_1, \ldots, x_r]$. Our proof of Noether normalization showed how to find $x_1', \ldots, x_r', x_{r+1}, \ldots, x_n$ so that $x_n$ is integral over $k[x_1', \ldots, x_r']$.

(a) If $k$ is infinite, then the $x_i'$ can be chosen to be $k$-linear combinations of $x_1, \ldots, x_r, x_n$.

(b) If $k$ is infinite, $R$ graded, $R_0 = k$ and $R$ generated by $R_1$, then the $x_i$ (in the conclusion - i.e. $R$ is integral over $k[x_1, \ldots, x_r]$ and $x_1, \ldots, x_n$ generate $R$) can be chosen to be in $R_1$.

19. (10/05) State and prove a uniqueness theorem for the valuation associated to a valuation ring.

20. (10/24)

(a) Let $R$ be a PID which is not a field. Then $\dim R[x] = 2$.

(b) Let $k$ be algebraically closed and $f$ an irreducible polynomial in $R = k[x, y]$. The $R/(f)$ is a Dedekind ring if and only if $f$ is nonsingular.

21. (10/29) Let $R$ be a Noetherian ring. Then $\dim R[x] = \dim R + 1$.

22. (10/29) For any $R$, $\dim R[x] \leq 1 + 2 \dim R$.

23. (10/31) - Alternative to Problem 20b Let $f$ be an irreducible polynomial in $k[x_1, \ldots, x_n], k$ algebraically closed and let $R = k[x_1, \ldots, x_n]/(f)$. Prove that the following are equivalent:
   (1) For every maximal ideal $\mathfrak{m}$ of $R$, $R_{\mathfrak{m}}$ is a regular local ring.
   (2) $f$ is nonsingular.
24. (11/02) Let $R$ be a standard graded Noetherian $k$-algebra, $k$ a field. Define the graded dimension of $R$ to be the length of a maximal chain of graded prime ideals. Prove that the graded dimension of $R$ equals the usual dimension of $R$. Corollary - the order of the pole of one for the Hilbert series of $R$ equals the dimension of $R$.

25. (11/09) Let $k$ be a field and $\Delta$ be an $d$-dimensional simplicial complex with vertices $\{1, \ldots, n\} = [n]$. The face ring, also known as the Stanley-Reisner ring, of $\Delta$ is

$$k[\Delta] = k[x_1, \ldots, x_n]/I_\Delta,$$

$$I_\Delta = <x_{i_1} \cdots x_{i_m} : \{i_1, \ldots, i_m\} \notin \Delta>.$$

Prove that $\dim k[\Delta] = d + 1$.

26. (11/14) Let $R$ be a Noetherian local ring and $M$ a f.g. $R$-module. Then $\text{depth } M = \text{depth } \hat{M}$.

27. (11/16) Let $R$ be a local Noetherian ring and $M$ a f.g. $R$-module. Let $P$ be an associated prime of $M$ and $I \subseteq P$. Prove that the annihilator of $M/IM$ is contained in $P$.

Optional addition: What is the best theorem you can prove along these lines?

28. (11/19) T/F. Let $M$ be a f.g. Cohen-Macaulay module over a Noetherian ring $R$. If $(x_1, \ldots, x_n)$ is a maximal regular sequence for $M$, then for all $1 \leq i \leq n$ the quotient module $M/(x_1, \ldots, x_i)M$ is Cohen-Macaulay.

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