Math 4550 Solution to 5 from Feb. 24, 2009

Prove that Stanley's trick works as a method for computing h_i.

It is possible to do this directly by induction on d. However, the following might make it clearer what is going on. Let's look at an example to better understand what is happening. Suppose the f-vector (including f_{-1}) is $(1, 7, 14, 9)$. Stanley's trick says that we can compute the $h's$ as follows.

$$\begin{array}{ccccccccc}
&&&&1&&&& \\
&&&1&&7&&& \\
&&1&&6&&14&& \\
&1&&5&&8&&9& \\
1&&4&&3&&1&&
\end{array}.$$

So $h_0 = 1, h_1 = 4, h_2 = 3$ and $h_3 = 1$. Look what happens if we rotate the triangle one time clockwise so that the 1's are now on the right side and the h's are on the left side.

$$\begin{array}{cccccccc}
&&&1&&&& \\
&&4&&1&&& \\
&3&&5&&1&& \\
1&&8&&6&&1& \\
&9&&14&&7&&1
\end{array}. \qquad (1)$$

Here, we just have Pascal's triangle with different values, the h's, along the left-hand side and the f's in reverse order along the bottom. So, the problem is equivalent to showing that this operation - but 1's on the r.h.s, the h's along the l.h.s and fill in Pascal's triangle gives us the f's along the bottom in reverse order. This is surprisingly simple to see. Look at the Pascal's triangles below:

$$\begin{array}{cccccccc}
&&&1&&&& \\
&&0&&1&&& \\
&0&&1&&1&& \\
0&&1&&2&&1& \\
&1&&3&&3&&1
\end{array}. \quad
\begin{array}{cccccccc}
&&&0&&&& \\
&&4&&0&&& \\
&0&&4&&0&& \\
0&&4&&4&&0& \\
&4&&8&&4&&0
\end{array}. \quad
\begin{array}{cccccccc}
&&&0&&&& \\
&&0&&0&&& \\
&3&&0&&0&& \\
0&&3&&0&&0& \\
&3&&3&&0&&0
\end{array}. \quad
\begin{array}{cccccccc}
&&&0&&&& \\
&&0&&0&&& \\
&0&&0&&0&& \\
1&&0&&0&&0& \\
&1&&0&&0&&0
\end{array}.$$

Our triangle (1) is the sum of the four above triangles. In fact, we can easily see that the four triangles above represent the contribution of $h_0 = 1, h_1 = 4, h_2 = 3$ and $h_3 = 1$ in (1).

What about the general case? Our triangle will have to have $d + 1$ rows.

$$\begin{array}{ccccccccccc}
&&&&&1&&&&& \\
&&&&h_1&&1&&&& \\
&&&h_2&&h_1+1&&1&&& \\
&&h_3&&h_2+h_1+1&&h_1+2&&1&& \\
&h_4&&h_3+h_2+h_1+1&&h_2+2h_1+3&&h_1+3&&1& \\
h_5&&h_4+h_3+h_2+h_1+1&&h_3+2h_2+3h_1+4&&h_2+3h_1+6&&h_1+4&&1 \\
&\vdots&&\vdots&&&\vdots&&\vdots&&
\end{array},$$

and be the sum of these triangles.

$$\begin{array}{ccccccccccc}
&&&&&1&&&&& \\
&&&&0&&1&&&& \\
&&&0&&1&&1&&& \\
&&0&&1&&2&&1&& \\
&0&&1&&3&&3&&1& \\
0&&1&&4&&6&&4&&1 \\
&\vdots&&\vdots&&&\vdots&&\vdots&&
\end{array}
\quad
\begin{array}{ccccccccccc}
&&&&&0&&&&& \\
&&&&h_1&&0&&&& \\
&&&0&&h_1&&0&&& \\
&&0&&h_1&&h_1&&0&& \\
&0&&h_1&&2h_1&&h_1&&0& \\
0&&h_1&&3h_1&&3h_1&&h_1&&0 \\
&\vdots&&\vdots&&&\vdots&&\vdots&&
\end{array},$$

$$
\begin{array}{c}
0 \\
0 \quad 0 \\
h_2 \quad 0 \quad 0 \\
0 \quad h_2 \quad 0 \quad 0 \\
0 \quad h_2 \quad h_2 \quad 0 \quad 0 \\
0 \quad h_2 \quad 2h_2 \quad h_2 \quad 0 \quad 0 \\
\vdots \quad \vdots \qquad \vdots \quad \vdots
\end{array}
\quad , \quad
\begin{array}{c}
0 \\
0 \quad 0 \\
0 \quad 0 \quad 0 \\
h_3 \quad 0 \quad 0 \quad 0 \\
0 \quad h_3 \quad 0 \quad 0 \quad 0 \\
0 \quad h_3 \quad h_3 \quad 0 \quad 0 \quad 0 \\
\vdots \quad \vdots \qquad \vdots \quad \vdots
\end{array}
\quad ,
$$

etc.

Now look at the last row. The last entry is supposed to be $f_{-1} = 1$. The only Pascal's triangle which is nonzero there is the first one - so it is in fact 1. What about the next to last entry, which is supposed to be f_0? The only Pascal's triangles which are nonzero there are the first two. In the first one it (the next to last entry on the $(d+1)^{st}$ row) is the next to last entry on the d^{th} row of the usual Pascal's triangle, which equals $\binom{d}{1}$, and the last entry of the second triangle - which is h_1 times the usual Pascal's triangle with $d-1$ rows $= h_1 \cdot 1 = h_1$. So, this entry is $h_1 + \binom{d}{1}$ as expected. One more to see what is going on. The third to last entry on the last row is nonzero in the first three Pascal's triangles. In the first one it is the third to last entry in a Pascal's triangle with d rows $= \binom{d}{2}$. In the second one it is the next to last entry in h_1 times a Pascal's triangle with $d-1$ rows, $= h_1 \cdot \binom{d-1}{1}$. In the third one it is the last entry of h_2 times a Pascal's triangle with $d-2$ rows $= h_2 \cdot \binom{d-2}{0}$. All told it is $h_2 + \binom{d-1}{1}h_1 + \binom{d}{2}$. If you check the formula derived for f_1 you will see that this checks - and that the appropriate pattern continues for all the f_i's.