

Math 4550 Solution to 6 from Feb. 3, 2009

(6.) Prove that the convex hull of $A \subseteq \mathbb{R}^d$ is the set of all convex combinations of elements of A .

Proof: Let Z be the set of all convex combinations of elements of A . First we show that Z is convex, hence $ch(A)$ is contained in Z since it is the intersection of all convex sets containing A . Let $x, y \in Z$ and write

$$x = \sum_{i=1}^n c_i x_i,$$
$$y = \sum_{i=1}^n b_i x_i,$$

where the x_i are in A , $c_1 + \cdots + c_n = b_1 + \cdots + b_n = 1$ and all $c_i \geq 0, b_i \geq 0$. We can assume that both expressions use the same points of A by using extra zeros for coefficients. Then,

$$tx + (1-t)y = \sum_{i=1}^n [tc_i + (1-t)b_i]x_i.$$

Each term $tc_i + (1-t)b_i$ is greater than or equal to zero since each c_i and b_i is, and the sum of all the coefficients is $t(c_1 + \cdots + c_n) + (1-t)(b_1 + \cdots + b_n) = 1$.

Now we show that Z is contained in the convex hull of A . We do this by showing Z is contained in any convex set which contains A . This is done by induction on n , which by definition begins with $n = 2$, where the assertion holds by definition. Now consider a convex combination

$$x = \sum_{i=1}^n c_i x_i.$$

If $c_n = 1$, then all the other c_i are zero and $x = x_n$ is in A . Otherwise,

$$x = c_n x_n + (1 - c_n) \left[\frac{1}{1 - c_n} \sum_{i=1}^{n-1} c_i x_i \right]. \tag{1}$$

Since $c_1 + \cdots + c_{n-1} = 1 - c_n$ the sum

$$\frac{1}{1 - c_n} \sum_{i=1}^{n-1} c_i x_i$$

is a convex combination of elements of A with only $n - 1$ terms. Hence, by the induction hypothesis, it is in any convex set containing A . Now the right-hand side of (1) is on a line segment through two points of a convex set.