Solution to #4, hw 2, 2009

Let $x \in K$ and write $x = c_1x_1 + \cdots + c_mx_m$ as a convex combination of the elements of X with m as small as possible. In this case the x_i must be distinct and all of the c_i are nonzero. If $m \leq d+1$ we are done. So assume that $m \geq d+2$. This implies that $x_1, \ldots, x_m$ are affinely dependent. So there exists a nontrivial affine dependence $b_1x_1 + \cdots + b_mx_m = \vec{0}$ with $b_1 + \cdots + b_m = 0$. By reordering, and possibly multiplying this dependence by -1, we can assume that $b_m > 0$ and $c_m/b_m \leq c_i/b_i$ whenever $b_i > 0$. Define $a_i = c_i - \frac{c_m}{b_m}b_i$. Now,

$$\sum a_i = \sum c_i - \frac{c_m}{b_m}\sum b_i = 1 - 0 = 1.$$

Furthermore, each $a_i \geq 0$. Indeed, if $b_i \leq 0$, then, since $\frac{c_m}{b_m} \geq 0, a_i \geq c_i \geq 0$. If $b_i > 0$, then, since $c_i/b_i \geq c_m/b_m$, we again see that $a_i \geq 0$. Repeating the above calculation with the x_i,

$$\sum a_ix_i = \sum c_ix_i - \frac{c_m}{b_m}\sum b_ix_i = x - \vec{0} = x.$$

But this is a contradiction as $a_m = 0$ and we can now write x as a convex combination of elements of X with only $m-1$ terms.